# Taylor approximation is not a universal problem solver

Seewoo Lee

**Abstract**

This is an example where Taylor approximation does not work at all.

Define a function $y = f(x)$ as follows:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

**Q1.** Show that the function is continuous.

**Q2.** Show that the function is differentiable. What is $f'(0)$? To compute this, you can use the following fact without proof: exponential grows faster than any polynomials. In other words, for any $N \geq 0$,

$$\lim_{x \to \infty} \frac{x^N}{e^x} = 0$$

**Q3.** For $x \neq 0$, show that $f^{(n)}(x)$ has a form of

$$f^{(n)}(x) = e^{-1/x^2} p_n(1/x)$$

for a polynomial $p_n(x)$. (Hint: use mathematical induction on $n$ to find a recursive relation on $p_n$.) Can you express degree of $p_n$ in terms of $n$?

**Q4.** Using Q3, show that $f^{(n)}(0) = 0$ for any $n \geq 1$.

**Q5.** What is $n$-th Taylor polynomial of $f(x)$ at $x = 0$? (Answer: zero)

Hence for this specific function, the Taylor approximations are zero and they do not give you any useful approximations of $e^{-1/x^2}$. The functions where $\lim_{n \to \infty} T_n f(x) = f(x)$ hold are called *real analytic functions*, and the above function is an example of non-real analytic function. Most of the functions we see in the course are real analytic though.