Why matrix multiplication is weird?

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Abstract

This is an exercise that explains what *is* a matrix multiplication and *why* it is defined in that way.

You are playing a new RPG game called **Math10A**. You have two abilities - **HP (Homework Point)** and **MP (Midterm Point)** - determine the strengths of your skills - **Heavy Vectors** and **Infinitesimal Shift**. The following table shows the strengths of skills based on each ability point (Table 1).

	HP	MP
Heavy Vectors	4	1
Infinitesimal Shift	3	6

Table 1: ability-to-skill matrix

Q1. If your HP and MP are 300 and 200, respectively, what are the strengths of each skill above? Compare your answer with the following matrix-vector multiplication

$$\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} \times \begin{pmatrix} 300 \\ 200 \end{pmatrix}.$$

In the Math10A world, there are two monsters named **Eigen** and **Lim**. The following table shows the damage that each skill can give to each monster (Table 2). For example, Heavy Vectors give 30 damages to Eigen.

	Heavy Vectors	Infinitesimal Shift
Eigen	30	70
Lim	40	60

Table 2: skill-to-damage matrix

Q2. If you have strengths computed as above in Q1, what total damages can you give each monster, assuming that you're using each skill once for each monster? Compare your answer with the following matrix-vector multiplication

$$\begin{pmatrix} 30 & 70 \\ 40 & 60 \end{pmatrix} \times \begin{pmatrix} 1400 \\ 2100 \end{pmatrix}.$$

Q3. Compute

$$\begin{pmatrix} 30 & 70 \\ 40 & 60 \end{pmatrix} \times \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$$

and

$$\begin{pmatrix} 30 & 70 \\ 40 & 60 \end{pmatrix} \times \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} \times \begin{pmatrix} 300 \\ 200 \end{pmatrix}.$$

Did you get the same answer?

Q4. If you compute matrix multiplication *in a wrong way* as

$$\begin{pmatrix} 30 & 70 \\ 40 & 60 \end{pmatrix} \times \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} " = " \begin{pmatrix} 30 \times 4 & 70 \times 1 \\ 40 \times 3 & 60 \times 6 \end{pmatrix},$$

do you still get the same answer?

Q5. If there are 5 types of points, 3 types of skills, and 4 types of monsters, what will be the dimensions of corresponding matrices? How about p points, s skills, and m monsters? Can you still do the *wrong* multiplication?

This exercise implicitly tells you that matrices are the ways of representing *linear transforms* as *array of numbers*, and the *multiplication of matrices* corresponds to *composition of linear transforms*. That's why the multiplication is defined in such a weird-looking way.