1. Compute the determinant of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix}$$

2. Is A invertible? What about B and C?

3. Compute the inverse of the invertible matrices.

4. Find a solution to the following system of linear equations.

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 4x_2 & = & -2 \end{array}$$

5. For which values of a does the following system of linear equations have a unique solution?

$$\begin{array}{rcrcl} 2x_1 & + & 6x_2 & = & \pi \\ 3x_1 & + & ax_2 & = & \sqrt{2} \end{array}$$

6. Which of the following vectors are eigenvectors of B?

$$\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

7. What are the eigenvalues of B?

8. What is the characteristic polynomial of the following matrix?

$$D = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

9. What are the eigenvalues of D?

1. Compute the determinant of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2 \\ \det(B) &= (-1) \cdot (-6) - 2 \cdot 3 = 6 - 6 = 0 \\ \det(C) &= 0 \cdot 4 \cdot 9 + 1 \cdot 5 \cdot 6 + 2 \cdot 3 \cdot 7 - 2 \cdot 4 \cdot 6 - 0 \cdot 5 \cdot 7 - 1 \cdot 3 \cdot 9 \\ &= 0 + 30 + 42 - 48 - 0 - 27 \\ &= -3 \end{aligned}$$

2. Is A invertible? What about B and C?

   Since $\det(A) \neq 0$, it is invertible. Since $\det(B) = 0$, it is not invertible. Since $\det(C) \neq 0$, it is invertible.

3. Compute the inverse of the invertible matrices.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Since

$$\left(C \mid I\right) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{I}\leftrightarrow\text{II}} \left(\begin{array}{ccc|ccc} 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{III}-2\text{I}} \left(\begin{array}{ccc|ccc} 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right) \xrightarrow{\text{III}+\text{II}} \left(\begin{array}{ccc|ccc} 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{\text{II}-2\text{III}} \left(\begin{array}{ccc|ccc} 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{\text{I}-5\text{III}} \left(\begin{array}{ccc|ccc} 3 & 4 & 0 & -5 & 11 & -5 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{\text{I}-4\text{II}} \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & -5 & 3 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{\text{I}\cdot\frac{1}{3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{5}{3} & 1 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right),$$

we have

$$C^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{5}{3} & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

4. Find a solution to the following system of linear equations.

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 4x_2 & = & -2 \end{array}$$

The system of linear equations is equivalent to

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Since A is invertible, the only solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

5. For which values of a does the following system of linear equations have a unique solution?

$$\begin{array}{rcrcl} 2x_1 & + & 6x_2 & = & \pi \\ 3x_1 & + & ax_2 & = & \sqrt{2} \end{array}$$

The system of linear equations has a unique solution if

$$0 \neq \det \begin{bmatrix} 2 & 6 \\ 3 & a \end{bmatrix} = 2a - 18,$$

i.e. if $a \neq 9$.

6. Which of the following vectors are eigenvectors of B?

$$B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Multiply B to each vector and see if the result is parallel to the vector.

$$B\mathbf{v} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = -7\mathbf{v}$$
$$B\mathbf{w} = \begin{bmatrix} 5 \\ -15 \end{bmatrix} \nparallel \mathbf{w}$$
$$B\mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} \nparallel \mathbf{x}$$
$$B\mathbf{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\mathbf{y}$$
$$B\mathbf{z} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \nparallel \mathbf{z}$$

(Here the symbol $\nparallel$ means that two vectors are not parallel.) Hence $\mathbf{v}$ and $\mathbf{y}$ are eigenvectors, with eigenvalues -7 and 0 respectively.

7. What are the eigenvalues of B?

As we saw above, eigenvalues would be -7 and 0. One can find this by solving $\det(B - \lambda I) = 0$:

$$\begin{aligned} \det(B - \lambda I) &= \det\left(\begin{bmatrix} -1-\lambda & 2 \\ 3 & -6-\lambda \end{bmatrix}\right) = (-1-\lambda)(-6-\lambda) - 6 \\ &= \lambda^2 + 7\lambda = \lambda(\lambda+7) = 0 \\ &\Leftrightarrow \lambda = -7, 0 \end{aligned}$$

8. What is the characteristic polynomial of the following matrix?

$$D = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\det(D - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix}\right) = (1-\lambda)(-1-\lambda) - 2 = \lambda^2 - 3.$$