1. Find eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & \frac{1}{2} \\ -3 & -\frac{1}{2} \end{bmatrix}$$

2. Using 1, find an invertible matrix P and a digonal matrix D such that $A = PDP^{-1}$. Can you compute A^{100}?

3. Let B be the following 3 by 3 matrix.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

   Find the eigenvalues of B.

4. Check that the following vectors are eigenvectors of B. What are the corresponding eigenvalues?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

5. Let $D = (d_{ii})$ be a 3×3 diagonal real matrix whose entry in the $i-$th row and $i-$th column is d_{ii}. What are the eigenvalues of D?

6. Let A, B be $n \times n$ matrices, with A invertible. Is it possible for $A^{-1}BA$ to have different eigenvectors than B?

7. With A, B as above, is it possible for $A^{-1}BA$ to have different eigenvalues than B?

8. Find the eigenvalues and eigenvectors of

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

What's special about this matrix?

1. Find eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & \frac{1}{2} \\ -3 & -\frac{1}{2} \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & \frac{1}{2} \\ -3 & -\frac{1}{2}-\lambda \end{bmatrix}\right) = (2-\lambda)\left(-\frac{1}{2}-\lambda\right) - \frac{1}{2}(-3)$$
$$= \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right)$$

So the eigenvalues are $\lambda = 1, \frac{1}{2}$. The corresponding eigenvectors are

- $\lambda_1 = 1$: if $\mathbf{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$,

  $$(A - \lambda_1 I)\mathbf{v} = \left(\begin{bmatrix} 2 & \frac{1}{2} \\ -3 & -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ -3 & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \frac{1}{2}y \\ -3x - \frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  One of the two equations is redundant (they are constant multiple of each other), so $x + \frac{1}{2}y = 0 \Leftrightarrow y = -2x$. We can choose $x = 1, y = -2$, and get $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

- $\lambda_2 = \frac{1}{2}$: if $\mathbf{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix}$,

  $$(A - \lambda_2 I)\mathbf{v} = \left(\begin{bmatrix} 2 & \frac{1}{2} \\ -3 & -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x + \frac{1}{2}y \\ -3x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  One of the two equations is redundant (they are constant multiple of each other), so $\frac{3}{2}x + \frac{1}{2}y = 0 \Leftrightarrow y = -3x$. We can choose $x = 1, y = -3$, and get $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

2. Using 1, find an invertible matrix P and a digonal matrix D such that $A = PDP^{-1}$. Can you compute A^{100}?

Set

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Then we have $AP = PD \Leftrightarrow A = PDP^{-1}$ (this process is called *diagonalization*). Then

$$P^{-1} = \frac{1}{1 \cdot (-3) - 1 \cdot (-2)} \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} = PDP^{-1} \cdot PDP^{-1} \cdot \dots \cdot PDP^{-1} \\ &= PDD \cdots DP^{-1} \qquad \text{(all the } PP^{-1}\text{'s in the middle become identity matrices and disappear)} \\ &= PD^{100}P^{-1} \end{aligned}$$

Power of a diagonal matrix is a diagonal matrix with powers as entries:

$$D^{100} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{100} = \begin{bmatrix} 1^{100} & 0 \\ 0 & \left(\frac{1}{2}\right)^{100} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2^{100}} \end{bmatrix}$$

so we get

$$\begin{aligned} A^{100} = PD^{100}P^{-1} &= \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2^{100}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2^{100}} \\ -2 & -\frac{3}{2^{100}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 - \frac{1}{2^{99}} & 1 - \frac{1}{2^{100}} \\ -6 + \frac{6}{2^{100}} & -2 + \frac{3}{2^{100}} \end{bmatrix}. \end{aligned}$$

3. Let B be the following 3 by 3 matrix.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

Find the eigenvalues of B.

$$\det(B - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & -\lambda & 0 \\ 0 & -1 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(-\lambda)(3-\lambda) = 0$$

so $\lambda = 1, 0, 3$.

4. Check that the following vectors are eigenvectors of B. What are the corresponding eigenvalues?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

Multiply B to the vectors, we get

$$B\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 3\mathbf{u}$$

$$B\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{v}$$

$$B\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{w}$$

So they are eigenvectors of B corresponding to $3, 1, 0$ respectively.

5. Let $D = (d_{ii})$ be a 3×3 diagonal real matrix whose entry in the $i-$th row and $i-$th column is d_{ii}. What are the eigenvalues of D?

We have

$$\det(D - \lambda I) = \det\left(\begin{bmatrix} d_{11} - \lambda & 0 & 0 \\ 0 & d_{22} - \lambda & 0 \\ 0 & 0 & d_{33} - \lambda \end{bmatrix}\right) = (d_{11} - \lambda)(d_{22} - \lambda)(d_{33} - \lambda) = 0$$

hence the eigenvalues are just the digonal entries $\lambda = d_{11}, d_{22}, d_{33}$. This is true for any $n \times n$ matrices in general.

6. Let A, B be $n \times n$ matrices, with A invertible. Is it possible for $A^{-1}BA$ to have different eigenvectors than B?

7. With A, B as above, is it possible for $A^{-1}BA$ to have different eigenvalues than B?

Let's do 6 and 7 together. Let's say $(\lambda, \mathbf{v})$ is eigenvalue and eigenvector for B and $(\mu, \mathbf{w})$ is eigenvalue and eigenvector for $A^{-1}BA$. This means that we have equations

$$B\mathbf{v} = \lambda\mathbf{v}, \quad A^{-1}BA\mathbf{w} = \mu\mathbf{w}.$$

By multiplying A^{-1} to the first equation, we get

$$A^{-1}B\mathbf{v} = A^{-1}\lambda\mathbf{v} \Leftrightarrow (A^{-1}BA)(A^{-1}\mathbf{v}) = \lambda(A^{-1}\mathbf{v})$$

Since $\mathbf{v} \neq \mathbf{0}$, we should have $A^{-1}\mathbf{v} \neq \mathbf{0}$ - if $A^{-1}\mathbf{v} = \mathbf{0}$, multiplying A on both sides gives $\mathbf{v} = \mathbf{0}$. Hence λ is also an eigenvalue of $A^{-1}BA$ with an eigenvector $A^{-1}\mathbf{v}$. Similarly, from the second equation, multiplying A gives

$$B(A\mathbf{w}) = AA^{-1}BA\mathbf{w} = A(\mu\mathbf{w}) = \mu(A\mathbf{w}).$$

By a similar argument as above, $\mathbf{w} \neq \mathbf{0}$ implies $A\mathbf{w} \neq \mathbf{0}$, so μ also becomes an eigenvalue of B with an eigenvector $A\mathbf{w}$. So we conclude that the eigenvalues of B and $A^{-1}BA$ are the same (so the answer for 7 is impossible).

Regarding eigenvectors, note that if $\mathbf{v}$ is an eigenvector, any nonzero multiple of it is also an eigenvector for the same matrix and same eigenvalue. So the eigenvectors can be different (so the answer for 6 is possible).

8. Find the eigenvalues and eigenvectors of

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

What's special about this matrix?

As we usually do, we solve $\det(C - \lambda I) = 0$, which gives $\lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$. For each eigenvalues, the corresponding eigenvectors are (multiples of)

- $\lambda = i \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$
- $\lambda = -i \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

So the eigenvalues and eigenvectors are not real (they are complex numbers and vectors). In fact the matrix C represents a 90-degree counter-clockwise rotation of a vector (it sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} -y \\ x \end{bmatrix}$) so $C\mathbf{v}$ never be able to a constant multiple of $\mathbf{v}$ when $\mathbf{v}$ is a nonzero real vector.