1. Diagonalize the following matrices. In other words, for each matrix, find an invertible matrix P and a diagonal matrix D such that the matrix equals to PDP^{-1}.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

2. Compute A^{20}.

   Use the above digonalization.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{20} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{20} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{1}{2}(3^{20}-1) \\ 0 & 3^{20} \end{bmatrix}$$

3. Let

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   Compute C^2, C^3, C^4, and C^5. Can you find a pattern?

$$C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, \quad C^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad C^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = C.$$

   It is periodic with the period 4.

4. What is C^{2023}? Can you compute it using diagonalization?

   Since $2023 = 4 \times 505 + 3$, $C^{2023} = C^3$. Using diagonalization, you can check that $i, -i$ are eigenvalues of C with eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$, so

$$C = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

$$C^{2023} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{2023} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   (Note that $i = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \dots$)

5. Consider the recursion of the form

$$\mathbf{x}_{t+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{x}_t, \qquad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What is $\mathbf{x}_5$?

Let A be the above matrix and diagonalize it. We have

$$\det(A - \lambda I) = \lambda^2 - \frac{7}{6} + \frac{1}{6} = (\lambda - 1)\left(\lambda - \frac{7}{6}\right) \Rightarrow \lambda = 1, \frac{1}{6}.$$

(a) For $\lambda_1 = 1$, the corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ satisfies

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\frac{1}{2}x_1 + \frac{1}{2}y_1 = 0.$$

We can take $x_1 = 1$ and $y_1 = 1$, and we get an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) For $\lambda_2 = \frac{1}{6}$, the corresponding eigenvector $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ satisfies

$$(A - \lambda_2 I)\mathbf{v}_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{3}x_2 + \frac{1}{2}y_2 = 0.$$

We can take $x_2 = 3$ and $y_2 = -2$, and we get an eigenvector $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

So we have

$$A = PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}^{-1}$$

and

$$\mathbf{x}_5 = A^5\mathbf{x}_0 = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6^5} \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{3}{5\times 6^5} \\ \frac{2}{5} - \frac{2}{5\times 6^5} \end{bmatrix}$$

6. What is the limiting behavior of $\mathbf{x}_t$, i.e. $\mathbf{y} = \lim_{t\to\infty} \mathbf{x}_t$?

$$\mathbf{y} = \lim_{t\to\infty} \mathbf{x}_t = \lim_{t\to\infty} A^t\mathbf{x}_0 = \lim_{t\to\infty} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}^t \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \end{bmatrix}$$

7. Check that $\mathbf{y}$ satisfies

$$\mathbf{y} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{y}.$$

   Can you explain why?

   You can directly check it by multiplying the matrix. Intuitively, $\mathbf{y}$ is the limiting behavior of $\mathbf{x}_t$, so it is *stable* (equilibrium) in the sense that the state does not change for further steps.

8. Let A be a 3 by 3 matrix with the eigenvalues and eigenvectors.

$$\lambda_1 = 1, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0.7, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -0.3, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

   Find $\lim_{k\to\infty} A^k$.

   Based on given eigenvalues and eigenvectors, we can diagonalize A as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

Since $A^k = (PDP^{-1})^k = PD^kP^{-1}$, we have

$$\begin{aligned}
\lim_{k\to\infty} A^k &= P\left(\lim_{k\to\infty} D^k\right)P^{-1} \\
&= PDP^{-1} \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\lim_{k\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}^k \right) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\lim_{k\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7^k & 0 \\ 0 & 0 & 0.3^k \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$