1. Determine whether the following sequences converge or diverge. If it converges, find the limit.
   (a) $a_n = \frac{2n^2+1}{n^2+n+1}$
   (b) $a_n = \frac{n^4}{10n^3+100n^2+1}$
   (c) $a_n = \frac{2^n+3^n}{5^n}$
   (d) $a_n = n^3$
   (e) $a_n = \cos(n\pi/2)$

2. Determine whether the following limits exist or not. If it converges, find the limit.
   (a) $\lim_{x\to\infty} \frac{x^2+3}{3x^2-4}$
   (b) $\lim_{x\to\infty} \sqrt{9x^2+x} - 3x$
   (c) $\lim_{x\to\infty} \ln(x^4+x) - \ln(x^4)$
   (d) $\lim_{x\to 1} x/(x-1)$
   (e) $\lim_{x\to 1} (x^2-1)/(x-1)$
   (f) $\lim_{x\to 3} (\sqrt{x+6}-x)/(x^3-3x^2)$
   (g) $\lim_{x\to 0} \sin(2x)/x$
   (h) $\lim_{x\to \pi/4} \sin(2x)/x$
   (i) $\lim_{x\to\infty} \sin(2x)/x$

3. Give an example of a function $f(x)$ where
$$f(1), \quad \lim_{x\to 1^-} f(x), \quad \lim_{x\to 1^+} f(x)$$
all exists and finite but all different. Is $f(x)$ continuous?

4. (*) Consider the recursive sequence $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n})$. Compute the first few terms using calculators. Can you guess the limit? (Hint: compute a_n^2.) Show that your guess is correct assuming that the limit exists.

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(a) Converges to 2
(b) Diverges to ∞
(c) Converges to 0
(d) Diverges to ∞
(e) The sequence is periodic: $0, -1, 0, 1, 0, -1, 0, 1, \dots$, so the limit does not exist.

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(a) Converges to $1/3$
(b) Converges to $1/6$
(c) $\ln(x^4+x) - \ln(x^4) = \ln((x^4+x)/x^4)$, converges to $\ln(1) = 0$
(d) Limit does not exist.
(e) $\lim_{x\to 1} x+1 = 2$.

(f)
$$\frac{\sqrt{x+6}-x}{x^3-3x^2} = \frac{\sqrt{x+6}-x}{x^2(x-3)} = \frac{x+6-x^2}{x^2(x-3)}\frac{1}{\sqrt{x+6}+x} = -\frac{x+2}{x^2(\sqrt{x+6}+x)}$$
and this converges to $-5/54$ as $x \to 3$.

(g) $\lim_{x\to 0} 2 \cdot \sin(2x)/(2x) = 2$

(h) $\sin(2\pi/4)/(\pi/4) = 4/\pi$

(i) We have $-1/x \leq |\sin(2x)/x| \leq 1/x$ for $x > 0$, hence by the squeeze theorem we get $\lim_{x\to\infty} \sin(2x)/x = 0$.

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all exists and finite but all different. Is $f(x)$ continuous?

Define $f(x)$ as
$$f(x) = \begin{cases} 1 & x > 1 \\ 0 & x = 1 \\ -1 & x < 1 \end{cases}.$$
Then
$$f(1) = 0, \quad \lim_{x\to 1^-} f(x) = -1, \quad \lim_{x\to 1^+} f(x) = 1$$
and all of them are different. The function can't be continuous since the limits as $x \to 1$ and $f(1)$ are different (the limit even does not exist!)

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4. (*) Consider the recursive sequence $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n})$. Compute the first few terms using calculators. Can you guess the limit? (Hint: compute a_n^2.) Show that your guess is correct assuming that the limit exists.

First few terms are
$$1, 2, \frac{7}{4}, \frac{97}{56}, \ldots$$
and their squares are
$$1, 4, 3.0625, 3.0003, \ldots$$

which seems to converge to 3. This is the correct guess: if the limit is α, then it should satisfy

$$\alpha = \frac{1}{2}\left(\alpha + \frac{3}{\alpha}\right) \Leftrightarrow \alpha^2 = 3$$

and since $a_n > 0$ for all n, the only possible choice is $\alpha = \sqrt{3}$. This recursion comes from Newton–Raphson method and can be used to approximate square roots efficiently (once you replace 3 with another number, you get the square root of that number as a limit).