1. Which of the following functions are continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

$$g(x) = \begin{cases} 2 & \text{if } x = 1 \\ \frac{x^2+1}{x-1} & \text{if } x \neq 1 \end{cases}$$

$$h(x) = \begin{cases} e^{\sin(x)} & \text{if } x \leq \pi \\ \ln(x-\pi) & \text{if } x > \pi \end{cases}$$

$$i(x) = \begin{cases} \cos(x) & \text{if } x \leq 0 \\ x\sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

$$j(x) = \begin{cases} \frac{x}{\cos(x)} & \text{if } x < 0 \\ \sin(x) & \text{if } x = 0 \\ \ln(x+1) & \text{if } x > 0 \end{cases}$$

2. For each of the functions

$$f(x) = \frac{\ln(x+4)}{x^2-9}$$
$$g(x) = \frac{1}{4-2^{\frac{1}{x}}}$$
$$h(x) = \ln(1-\cos(x)),$$

answer the following.

(a) What is the function's domain?

(b) Is the function continuous on its domain?

3. Use the Intermediate Value Theorem to show that there is are x and y such that

(a) $\frac{\pi}{2} < x < \pi$ and $\ln(1-\cos(x)) = \frac{1}{2}$, and

(b) $e^{\sin(y)} = 2$.

You can use that $\frac{1}{2} < \ln(2) < 1$.

1. Which of the following functions are continuous on $(-\infty, \infty)$?

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Since

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x^3 = 1^3 = 1, \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^2 = 1^2 = 1, \text{ and} \\ f(1) &= 1^2 = 1, \end{aligned}$$

f is continuous at 1 and thus everywhere.

Since $g(x) = x + 1$ for all x, the function g is continuous.

Since

$$\begin{aligned} \lim_{x\to \pi^-} h(x) &= \lim_{x\to \pi^-} e^{\cos(x)} = e^{\sin(\pi)} = e^0 = 1 \text{ but} \\ \lim_{x\to \pi^+} h(x) &= \lim_{x\to \pi^+} \ln(x-\pi) = \lim_{x\to 0^+} \ln(x) = -\infty, \end{aligned}$$

h is discontinuous at 0.

Since $-x \leq x\sin\left(\frac{1}{x}\right) \leq x$ for $x > 0$ and $\lim_{x\to 0^+} -x = 0 = \lim_{x\to 0^+} x$, the Squeeze Theorem implies that $\lim_{x\to 0^+} i(x) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$. Since however $i(0) = \cos(0) = 1$, the function i is discontinuous at 0.

Since

$$\begin{aligned} \lim_{x\to 0^-} j(x) &= \lim_{x\to 0^-} \frac{x}{\cos(x)} = \frac{0}{\cos(0)} = \frac{0}{1} = 0, \\ \lim_{x\to 0^+} j(x) &= \lim_{x\to 0^+} \ln(x+1) = \ln(0+1) = 0, \text{ and} \\ j(0) &= \sin(0) = 0, \end{aligned}$$

j is continuous at 0 and thus everywhere.

2. For each of the functions

$$f(x) = \frac{\ln(x+4)}{x^2-9}$$
$$g(x) = \frac{1}{4-2^{\frac{1}{x}}}$$
$$h(x) = \ln(1-\cos(x)),$$

answer the following.

(a) What is the function's domain?

(b) Is the function continuous on its domain?

(a) The function $\ln(x+4)$ is defined if $x+4>0$, i.e. on $(-4,\infty)$. The function $\frac{1}{x^2-9}$ is defined if $x^2 \neq 9$, i.e. on $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. Thus the domain of f is $(-4,-3) \cup (-3,3) \cup (3,\infty)$.

The domain of g is $(-\infty,0) \cup \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},\infty\right)$.

The domain of h is $\cdots \cup (-2\pi,0) \cup (0,2\pi) \cup (2\pi,4\pi) \cup \dots$.

(b) Yes, by theorems 4 and 6 from chapter 2.5.

3. Use the Intermediate Value Theorem to show that there is are x and y such that

(a) $\frac{\pi}{2} < x < \pi$ and $\ln(1-\cos(x)) = \frac{1}{2}$, and

(b) $e^{\sin(y)} = 2$.

You can use that $\frac{1}{2} < \ln(2) < 1$.

(a) Note that $\ln(1-\cos(x))$ is continuous on $\left[\frac{\pi}{2},\pi\right]$ by the previous exercise. Since

$$\ln\left(1-\cos\left(\frac{\pi}{2}\right)\right) = \ln(1) = 0 < \frac{1}{2} < \ln(2) = \ln(1-\cos(\pi)),$$

the Intermediate Value Theorem implies that there is an x in $\left(\frac{\pi}{2},\pi\right)$ such that $\ln(1-\cos(x)) = \frac{1}{2}$.

(b) Note that $e^{\sin(y)}$ is continuous everywhere, thus in particular on $\left[0,\frac{\pi}{2}\right]$. Since

$$e^{\sin(0)} = e^0 = 1 < 2 < e = e^1 = e^{\sin\left(\frac{\pi}{2}\right)},$$

the Intermediate Value Theorem implies that there is a y in $\left(0,\frac{\pi}{2}\right)$ such that $e^{\sin(y)} = 2$.