1. Consider the function $f(x) = \frac{x^3}{3} - x$.
   (a) What are the critical points of f?
   (b) Does f have a local minimum?
   (c) Does f have a local maximum?

2. Consider the function $g(x) = \ln\left(|x| + \frac{1}{e}\right)$.
   (a) What is the domain of g?
   (b) What are the critical points of g?
   (c) Does g have an absolute minimum?
   (d) Does g have an absolute maximum?

3. Consider the function $h(x) = \sin\left(2\pi x - \frac{\pi}{3}\right) - \pi x$.
   (a) What are the critical points of h?
   You can use that $2\cos(x) = 1$ iff $x = 2\pi n \pm \frac{\pi}{3}$ for an integer n.
   (b) What are the local minima of h?
   (c) Does h have an absolute minimum?

4. Let $i : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $i''(x) < i'(x)$ for all x. Can i have a local minimum? If so, give an example of such a function.

5. Consider the function $j(x) = x^{-x}$ defined on $(0, \infty)$.
   (a) What are the critical points of j?
   (b) Does j have a local maximum?
   (c) Does j have a local minimum?
   (d) Does j have an absolute maximum?
   (e) Does j have an absolute minimum?

1. Consider the function $f(x) = \frac{x^3}{3} - x$.
   (a) What are the critical points of f?
   (b) Does f have a local minimum?
   (c) Does f have a local maximum?

   (a) Since $f'(x) = x^2 - 1 = (x+1)(x-1)$, the critical points are -1 and 1.
   (b) Since f' changes from negative to positive at 1, f has a local minimum at 1 by the first derivative test.
   (c) Since f' changes from positive to negative at -1, f has a local maximum at -1.

2. Consider the function $g(x) = \ln\left(|x| + \frac{1}{e}\right)$.
   (a) What is the domain of g?
   (b) What are the critical points of g?
   (c) Does g have an absolute minimum?
   (d) Does g have an absolute maximum?

   (a) Since $\ln(x)$ is defined for all $x > 0$, $g(x)$ is defined iff $|x| + \frac{1}{e} > 0$, which is the case for all x. Thus the domain of g is $(-\infty, \infty)$.
   (b) For $x < 0$, $g(x) = \ln\left(-x + \frac{1}{e}\right)$ and $g'(x) = -\frac{1}{-x+\frac{1}{e}}$. Thus $g'(x) \neq 0$ for all $x < 0$. For $x > 0$, $g(x) = \ln\left(x + \frac{1}{e}\right)$ and $g'(x) = \frac{1}{x+\frac{1}{e}}$. Thus $g'(x) \neq 0$ for all $x > 0$. Since $\lim_{x\to 0^-} g'(x) = -e$ but $\lim_{x\to 0^+} g'(x) = e$, the function g is not differentiable at 0. Thus 0 is the only critical point.
   (c) For all x, $g(x) \geq \ln\left(\frac{1}{e}\right) = -1$. Since $g(0) = -1$, the function g has an absolute minimum at 0.
   (d) Since g has an absolute minimum at its only critical point, g does not have an absolute maximum.

3. Consider the function $h(x) = \sin\left(2\pi x - \frac{\pi}{3}\right) - \pi x$.
   (a) What are the critical points of h?
   You can use that $2\cos(x) = 1$ iff $x = 2\pi n \pm \frac{\pi}{3}$ for an integer n.
   (b) What are the local minima of h?
   (c) Does h have an absolute minimum?
   (a) Since $0 = h'(x) = 2\pi\cos\left(2\pi x - \frac{\pi}{3}\right) - \pi$ iff $2\cos\left(2\pi x - \frac{\pi}{3}\right) = 1$ iff $2\pi x - \frac{\pi}{3} = 2\pi n \pm \frac{\pi}{3}$, the critical points are given by $x = \frac{1}{2\pi}\left(2\pi n + \frac{2\pi}{3}\right) = n + \frac{1}{3}$ and $x = \frac{2\pi n}{2\pi} = n$.
   (b) We have $h''(x) = -4\pi^2 \sin\left(2\pi x - \frac{\pi}{3}\right)$. Since

$$\begin{aligned} h''\left(n + \tfrac{1}{3}\right) &= -4\pi^2 \sin\left(2\pi n + \tfrac{\pi}{3}\right) \\ &= -4\pi^2 \sin\left(\tfrac{\pi}{3}\right) \\ &< 0, \end{aligned}$$

   h has a local maximum at $n + \frac{1}{3}$ by the second derivative test. Since

$$\begin{aligned} h''(n) &= -4\pi^2 \sin\left(2\pi n - \tfrac{\pi}{3}\right) \\ &= -4\pi^2 \sin\left(-\tfrac{\pi}{3}\right) \\ &> 0, \end{aligned}$$

   h has a local minimum at n by the second derivative test.
   (c) Since $h(n) = \sin\left(2\pi n - \frac{\pi}{3}\right) - \pi n = \sin\left(-\frac{\pi}{3}\right) - \pi n$, we have $h(0) > h(1) > h(2) > \ldots$. Thus h does not have an absolute minimum.

4. Let $i : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $i''(x) < i'(x)$ for all x. Can i have a local minimum? If so, give an example of such a function.

   No. Suppose i has a local minimum at a. Then a is a critical point, so $i'(a) = 0$. Thus $i''(a) < i'(a) = 0$, so i has a local maximum at a by the second derivative test.

5. Consider the function $j(x) = x^{-x}$ defined on $(0, \infty)$.
   (a) What are the critical points of j?
   (b) Does j have a local maximum?
   (c) Does j have a local minimum?
   (d) Does j have an absolute maximum?
   (e) Does j have an absolute minimum?

(a) We use logarithmic differentiation. Note that $\ln(j(x)) = -x\ln(x)$. Thus

$$\frac{j'(x)}{j(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln(j(x)) = \frac{\mathrm{d}}{\mathrm{d}x} - x\ln(x) = -\ln(x) - 1$$

Hence $j'(x) = j(x)\,(-\ln(x) - 1) = -x^{-x}\,(\ln(x) + 1)$. Since $j'(x) = 0$ iff $\ln(x) + 1 = 0$, the only critical point is $x = \frac{1}{e}$.

(b) For $x < \frac{1}{e}$, we have $\ln(x) < -1$, so $j'(x) > 0$. For $x > \frac{1}{e}$, we have $\ln(x) > -1$, so $j'(x) < 0$. Thus j' changes from positive to negative at $\frac{1}{e}$. Hence j has a local maximum at $\frac{1}{e}$ by the first derivative test.

(c) No, the only critical point corresponds to a local *maximum*.

(d) Since $\frac{1}{e}$ is a local maximum and there are no local minima, $\frac{1}{e}$ also has to be an absolute maximum.

(e) No, since there is not even a *local* minimum.