1. Let $f(x) = x\sqrt{1-x}$.
   (a) Find the domain of $f(x)$ and $f'(x)$.
   (b) Find all critical points of $f(x)$.
   (c) Find the intervals of increase or decrease.
   (d) Find the intervals of concavity and the inflection points.
   (e) Find the absolute maximum and minimum values on the interval $[-1, 1]$.

2. Compute the following limits.
   (a) $\lim_{x\to\infty} \frac{\ln x}{x}$
   (b) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$
   (c) $\lim_{x\to\infty} xe^{-x}$
   (d) $\lim_{x\to 0} xe^{-x}$
   (e) $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$
   (f) $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

3. (a) Find two numbers whose difference is 1 and whose product is a minimum.
   (b) Find two numbers whose product is 1 and whose sum is a minimum.
   (c) Find two positive numbers whose product is 1 and whose sum of squares is a minimum.

1. Let $f(x) = x\sqrt{1-x}$.
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   (a) By the product rule, $f'(x) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$. Domain $f(x)$ is $x \leq 1$, but of $f'(x)$ is $x < 1$.
   (b) If we solve $f'(x) = 0$, we get:

$$\sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = 0 \Leftrightarrow \sqrt{1-x} = \frac{x}{2\sqrt{1-x}} \Leftrightarrow 2(1-x) = x \Leftrightarrow x = \frac{2}{3}.$$

   So the unique critical point is $(\frac{2}{3}, f\left(\frac{2}{3}\right)) = (\frac{2}{3}, \frac{2}{3\sqrt{3}})$. Note that $x = 1$ is not a critical point since it is not in a domain of $f(x)$.
   (c) We have

$$f'(x) = \frac{2(1-x) - x}{2\sqrt{1-x}} = \frac{2-3x}{\sqrt{1-x}},$$

   and this is positive (resp. negative) if $x < 2/3$ (resp. $2/3 < x < 1$). Hence $f(x)$ increases on $(-\infty, 2/3)$ and decreases on $(2/3, 1)$.
   (d) The second derivative of $f(x)$ is

$$f''(x) = \frac{3x-4}{4(1-x)^{3/2}},$$

   and this is always negative on the domain $x < 1$. Hence $f(x)$ is concave downward for $x < 1$. There's no inflection point since $f''(x)$ is always negative on its domain.
   (e) Compare the boundary values $f(-1) = -\sqrt{2}$, $f(1) = 0$ with critical value $f(2/3) = 2/(3\sqrt{3})$. The absolute maximum is $f(2/3) = 2/(3\sqrt{3})$ and the absolute minimum is $f(-1) = -\sqrt{2}$.

2. Compute the following limits.
   (a) $\lim_{x\to\infty} \frac{\ln x}{x}$
   (b) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

(c) $\lim_{x\to\infty} xe^{-x}$

(d) $\lim_{x\to 0} xe^{-x}$

(e) $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$

(f) $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

(a) Since it has a form of ∞/∞, we can apply l'Hospical's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

(b) Since it has a form of $0/0$, we can apply l'Hospital's rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{4/(1+(4x)^2)} = \frac{1}{4}.$$

(c) Since the limit $\lim_{x\to\infty} \frac{x}{e^x}$ has a form of ∞/∞, we can apply l'Hospital's rule:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

(d) We don't need l'Hospital's rule (and actually we can't). The limit is $0 \cdot e^{-0} = 0$.

(e) Since it has a form of $0/0$, we can apply l'Hospital's rule:

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2} = \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\cos(2x)}{1} = 2$$

where we use l'Hospital's rule again for the last equality (actually we don't have to - can you interpret it as a derivative of some function?).

(f) We can write limit as

$$\lim_{x\to 1^+} \frac{x}{x-1} - \frac{1}{\ln x} = \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x}$$

which is the case where we can use l'Hospital's rule $(0/0)$. By applying it twice,

$$\lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\ln x + 1 - 1}{\ln x + (x-1)/x} = \lim_{x\to 1^+} \frac{\ln x}{\ln x + 1 - 1/x}$$
$$= \lim_{x\to 1^+} \frac{1/x}{1/x + 1/x^2} = \frac{1}{2}.$$

3. (a) Find two numbers whose difference is 1 and whose product is a minimum.

(b) Find two **positive** numbers whose product is 1 and whose sum is a minimum.

(c) Find two positive numbers whose product is 1 and whose sum of squares is a minimum.

(a) Let x, y be two numbers with $y - x = 1$. Then the product xy can be expressed as a function in x by $xy = x(x+1) = f(x)$. This function attains absolute minimum at $x = -1/2$ with $f(-1/2) = -1/4$. Hence two numbers are $-1/2$ and $1/2$.

(b) Let x, y be two positive numbers with $xy = 1$. Then the sum $x+y$ can be expressed as a function in x by $x + y = x + 1/x = f(x)$. Solving $f'(x) = 1 - 1/x^2 = 0$ gives $x = 1$ (since x is assumed to be positive) and the function attains absolute minimum at $x = 1$ with $f(1) = 2$. Hence two numbers are 1 and 1.

(c) Let x, y be two positive numbers with $xy = 1$. The sum of squares $x^2 + y^2$ can be expressed as a function in x by $x^2 + y^2 = x^2 + 1/x^2 = f(x)$. Solving $f'(x) = 2x - 2/x^3 = 0$ gives $x = 1$, and the function attains absolute minimum at $x = 1$ with $f(1) = 2$. Hence two numbers are 1 and 1.