1. Can you use L'Hôpital's Rule to compute the following limits? Compute the limit.
   (a) $\lim_{x\to 0} \frac{e^x-1}{x^2+x}$
   (b) $\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)}$
   (c) $\lim_{x\to 0^+} \frac{\cos(x)}{2x}$
   (d) $\lim_{x\to 0^+} \frac{\sin(x)}{x^2}$
   (e) $\lim_{x\to \infty} \frac{x^2}{e^x}$
   (f) $\lim_{x\to 0^+} x\ln(x)$
   (g) $\lim_{x\to \infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$

2. Assume that $\lim_{x\to\infty} f(x) + f'(x)$ converges and $\lim_{x\to\infty} e^x f(x) = \infty$. Show that $\lim_{x\to\infty} f'(x) = 0$.
   Hint: Look at $\frac{e^x f(x)}{e^x}$.

3. Due to a shortage in electric equipment required for building wind turbines, their cost $\frac{n^2}{32{,}000}$ in million dollars is not linear in the number n of wind turbines built. However every coal power plant costs 100 million dollars. Assume that a wind turbine produces 5 million kWh per year and a coal power plant a billion kWh. How many wind turbines and coal power plants should be built to satisfy a yearly electrity demand of 100,000 kWh at the lowest possible cost?

4. Find antiderivatives of the following functions.
   (a) $f(x) = \frac{e^{2x}+1}{e^x}$
   (b) $g(x) = \frac{e^x}{e^{2x}+1}$
   (c) $h(x) = \sqrt{x}\left(3\frac{\sqrt{x}}{x} - 5x^2\right)$
   (d) $i(x) = \sin(4x)$

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   (a) $\lim_{x\to 0} \frac{e^x-1}{x^2+x} = \lim_{x\to 0} \frac{e^x}{2x+1} = \frac{e^0}{2\cdot 0+1} = 1$
   (b) Since $\lim_{x\to 0} 2\sin(x) - \sin(2x) = 2\sin(0) - \sin(2\cdot 0) = 0$ and $\lim_{x\to 0} 1 - \cos(x) = 1 - \cos(0) = 0$, L'Hôpital's Rule applies. Thus $\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)} = \lim_{x\to 0} \frac{2\cos(x)-2\cos(2x)}{1-\cos(x)}$. We can apply L'Hôpital's Rule another two times to obtain $\lim_{x\to 0} \frac{2\cos(x)-2\cos(2x)}{1-\cos(x)} = \lim_{x\to 0} \frac{-2\sin(x)+4\sin(2x)}{\sin(x)} = \lim_{x\to 0} \frac{-2\cos(x)+8\cos(2x)}{\cos(x)} = \frac{-2\cos(0)+8\cos(2\cdot 0)}{\cos(0)} = \frac{-2+8}{1} = 6$.
   (c) Note that $\frac{1}{4x} \le \frac{\cos(x)}{2x} \le \frac{1}{2x}$ for all $x \in \left(0, \frac{\pi}{3}\right)$ and $\lim_{x\to 0^+} \frac{1}{4x} = \infty = \lim_{x\to 0^+} \frac{1}{2x}$. Thus $\lim_{x\to 0^+} \frac{\cos(x)}{2x} = \infty$ by the Squeeze Theorem.
   (d) Since $\lim_{x\to 0^+} \sin(x) = \sin(0) = 0$ and $\lim_{x\to 0^+} x^2 = 0^2 = 0$, L'Hôpital's Rule applies. Thus $\lim_{x\to 0^+} \frac{\sin(x)}{x^2} = \lim_{x\to 0^+} \frac{\cos(x)}{2x} = \infty$.
   (e) We apply L'Hôpital's Rule twice to obtain $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$.
   (f) Since $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to 0^+} x^{-1} = \infty$, L'Hôpital's Rule applies to the fraction $\frac{\ln(x)}{x^{-1}}$. Thus $\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0$
   (g) We have $\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty} \frac{e^{2x}+1}{e^{2x}-1} = \lim_{x\to\infty} \frac{2e^{2x}}{2e^{2x}} = 1$ where the third equality uses L'Hôpital's Rule.

2. Assume that $\lim_{x\to\infty} f(x) + f'(x)$ converges and $\lim_{x\to\infty} e^x f(x) = \infty$. Show that $\lim_{x\to\infty} f'(x) = 0$.
   Hint: Look at $\frac{e^x f(x)}{e^x}$.

   We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to\infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x\to\infty} f(x) + f'(x)$ where the second equality uses L'Hôpital's Rule. Thus

$$\begin{aligned}\lim_{x\to\infty} f'(x) &= \left(\lim_{x\to\infty} f(x) + f'(x)\right) - \left(\lim_{x\to\infty} f(x)\right)\\ &= \left(\lim_{x\to\infty} f(x) + f'(x)\right) - \left(\lim_{x\to\infty} f(x) + f'(x)\right)\\ &= 0.\end{aligned}$$

3. Due to a shortage in electric equipment required for building wind turbines, their cost $\frac{n^2}{32{,}000}$ in million dollars is not linear in the number n of wind turbines built. However every coal power plant costs 100 million dollars. Assume that a wind turbine produces 5 million kWh per year and a coal power plant a billion kWh. How many wind turbines and coal power plants should be built to satisfy a yearly electrity demand of 100,000 kWh at the lowest possible cost?

   If we build n wind turbines then we need to build $m = \frac{100{,}000 - 5n}{1{,}000}$ coal power plants. The total cost would be $c(n) = \frac{n^2}{32{,}000} + 100m = \frac{n^2}{32{,}000} + \frac{100{,}000 - 5n}{10}$ million dollars. Since $c'(n) = \frac{n}{16{,}000} - \frac{1}{2}$ has a root at $n = \frac{16{,}000}{2} = 8,000$ and $c''(8,000) = \frac{1}{16{,}000} > 0$, the function c takes its minimum at $8,000$ by the second derivative test. Thus we should build 8,000 wind turbines and $\frac{100{,}000 - 5\cdot 8{,}000}{1{,}000} = 100 - 40 = 60$ coal power plants.

4. Find antiderivatives of the following functions.
   (a) $f(x) = \frac{e^{2x}+1}{e^x}$
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   (c) $h(x) = \sqrt{x}\left(3\frac{\sqrt{x}}{x} - 5x^2\right)$
   (d) $i(x) = \sin(4x)$

   (a) $F(x) = e^x - e^{-x}$
   (b) $G(x) = \tan^{-1}(e^x)$
   (c) $H(x) = 3x - 2x^{\frac{7}{2}}$
   (d) $I(x) = -\frac{1}{4}\cos(4x)$