1. Consider the function

$$f(x) = \frac{3x^2 + 6x + 2}{x^2 + 3x + 2}.$$

   (a) Perform long division to express $f(x)$ in the form of $P(x) + \frac{Q(x)}{R(x)}$ where the degree of Q is less than the degree of R.

   (b) Express $\frac{Q(x)}{R(x)}$ as a sum of partial fractions $\frac{A}{ax+b}$.

   (c) Compute $\int_0^1 f(x)\,\mathrm{d}x$.

2. Compute the partial fraction decomposition of

$$g(x) = \frac{1}{x^2 + 2x - 3}.$$

3. Find an antiderivative of
$$h(x) = \frac{x^4 + x^3 + x^2 + 1}{x^2 + x - 2}.$$

4. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.
   (a) $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$
   (b) $\int_{-\infty}^0 2^x\,dx$
   (c) $\int_{-\infty}^\infty x^3 - 3x^2\,dx$
   (d) $\int_0^1 x\ln(x)\,dx$

1. Consider the function

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   (b) Express $\frac{Q(x)}{R(x)}$ as a sum of partial fractions $\frac{A}{ax+b}$.

   (c) Compute $\int_0^1 f(x)\,\mathrm{d}x$.

   (a) $f(x) = 3 - \frac{3x+4}{x^2+3x+2}$

   (b) We have

$$\frac{3x+4}{x^2+3x+2} = \frac{3x+4}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

   for some constants A and B. Since $3x+4 = A(x+1)+B(x+2) = (A+B)x+A+2B$, we have $A+B=3$ and $A+2B=4$. Thus $A=2$ and $B=1$.

   (c) An antiderivative of $f(x)$ is

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= \int 3 - \frac{3x+4}{x^2+3x+2}\,\mathrm{d}x \\ &= \int 3\,\mathrm{d}x - \int \frac{2}{x+2}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x \\ &= 3x - 2\ln|x+2| - \ln|x+1|.\end{aligned}$$

   Thus $\int_0^1 f(x)\,\mathrm{d}x = 3 - 2\ln(3) - \ln(2) + \ln(2) + \ln(1) = 3 - 2\ln(3)$.

2. Compute the partial fraction decomposition of

$$g(x) = \frac{1}{x^2+2x-3}.$$

   Since $x^2+2x-3 = (x+3)(x-1)$, we have $g(x) = \frac{A}{x+3} + \frac{B}{x-1}$ for some constants A and B. Since $1 = A(x-1) + B(x+3) = (A+B)x - A + 3B$, we have $A+B=0$ and $-A+3B=1$. Thus $A = -\frac{1}{4}$ and $B = \frac{1}{4}$, i.e.

$$g(x) = -\frac{1}{4(x+3)} + \frac{1}{4(x-1)}.$$

3. Find an antiderivative of
$$h(x) = \frac{x^4 + x^3 + x^2 + 1}{x^2 + x - 2}.$$

We have $h(x) = x^2 + 3 + \frac{-3x+7}{(x+2)(x-1)} = x^2 + 3 + \frac{A}{x+2} + \frac{B}{x-1}$ for some constants A and B. Since $A(x-1) + b(x+2) = -3x + 7$, we have $A = \frac{-13}{3}$ and $B = \frac{4}{3}$. Hence

$$\int h(x)\,\mathrm{d}x = \frac{x^3}{3} + 3x - \frac{13}{3}\ln|x+2| + \frac{4}{3}\ln|x-1|.$$

4. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x$

(b) $\int_{-\infty}^0 2^x\,\mathrm{d}x$

(c) $\int_{-\infty}^\infty x^3 - 3x^2\,\mathrm{d}x$

(d) $\int_0^1 x\ln(x)\,\mathrm{d}x$

(a)

$$\begin{aligned}
\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^t (2x+1)^{-3}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \left.\frac{(2x+1)^{-2}}{-4}\right|_1^t \\
&= \lim_{t\to\infty} -\frac{1}{4}\left(\frac{1}{(2t+1)^2} - \frac{1}{3^2}\right) \\
&= -\frac{1}{4}\left(0 - \frac{1}{9}\right) \\
&= \frac{1}{36}
\end{aligned}$$

(b)

$$\begin{aligned}\int_{-\infty}^{0} 2^x \,\mathrm{d}x &= \lim_{t\to-\infty} \int_t^0 e^{\ln(2)x}\,\mathrm{d}x\\ &= \lim_{t\to-\infty} \left.\frac{2^x}{\ln(2)}\right|_t^0\\ &= \lim_{t\to-\infty} \frac{1}{\ln(2)}\left(1-2^t\right)\\ &= \frac{1}{\ln(2)}(1-0)\\ &= \frac{1}{\ln(2)}\end{aligned}$$

(c) Since

$$\begin{aligned}\int_{-\infty}^{\infty} x^3-3x^2\,\mathrm{d}x &= \int_{-\infty}^{0} x^3-3x^2\,\mathrm{d}x + \int_0^\infty x^3-3x^2\,\mathrm{d}x\\ &= \lim_{t\to-\infty}\int_t^0 x^3-3x^2\,\mathrm{d}x + \lim_{t\to\infty}\int_0^t x^3-3x^2\,\mathrm{d}x\\ &= \lim_{t\to-\infty} \left.\frac{x^4}{4}-x^3\right|_t^0 + \lim_{t\to\infty}\left.\frac{x^4}{4}-x^3\right|_0^t\\ &= -\infty+\infty,\end{aligned}$$

the integral is divergent.

(d) We obtain

$$\begin{aligned}\int_0^1 x\ln(x)\,\mathrm{d}x &= \lim_{t\to0^+}\int_t^1 x\ln(x)\,\mathrm{d}x\\ &= \lim_{t\to0^+} \left.\frac{x^2}{2}\ln(x)\right|_t^1 - \int_t^1 \frac{x}{2}\,\mathrm{d}x\\ &= \lim_{t\to0^+} 0-\frac{t^2\ln(t)}{2} - \left.\frac{x^2}{4}\right|_t^1\\ &= \lim_{t\to0^+} -\frac{t^2\ln(t)}{2}-\frac{1}{4}+\frac{t^2}{4}\\ &= -\frac{1}{4}\end{aligned}$$

using integration by parts, and L'Hôpital's Rule to compute

$$\lim_{t0^+} t^2\ln(t) = \lim_{t\to0^+}\frac{\ln(t)}{\frac{1}{t^2}} = \lim_{t\to0^+}\frac{\frac{1}{t}}{\frac{-2}{t^3}} = \lim_{t\to0^+} -\frac{t^2}{2} = 0.$$