1. Find the average value of the function $f(x) = x^2\sqrt{1+x^3}$ on the interval $[0, 2]$. 🍁

2. For constants a, b, consider the following equation of *ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

   Find the area of the region enclosed by the curve. (Hint: the curve looks like a squashed circle. Try to express the half of the area as an integral.) 🍁

3. Find the number c such that the average value of $f(x) = x^3 + x$ on the interval $[-1, c]$ is 0. 🍁

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The average value is

$$\frac{1}{2-0}\int_0^2 x^2\sqrt{1+x^3}dx = \frac{1}{2}\int_0^8 \frac{1}{3}\sqrt{1+u}du = \frac{1}{6}\left[\frac{2}{3}(1+u)^{3/2}\right]_0^8 = \frac{26}{9}$$

where we used the substitution $u = x^3$.

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Assume $a, b > 0$. The equation for the upper half of the curve is

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

which meets with the x-axis at $(-a, 0)$ and $(a, 0)$. Then the area of the upper half of the ellipse is

$$\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}dx = b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot a\cos\theta d\theta = ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta = ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta$$
$$= ab\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi ab}{2}$$

where we used the substitution $x = a\sin\theta$. Hence the total area is $2 \cdot \frac{\pi ab}{2} = \pi ab$.

3. Find the number c such that the average value of $f(x) = x^3 + x$ on the interval $[-1, c]$ is 0. 🍁

The average value can be expressed as

$$\frac{1}{c-(-1)}\int_{-1}^{c}(x^3+x)dx = \frac{1}{c+1}\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_{-1}^{c} = \frac{1}{c+1}\left(\frac{c^4}{4}+\frac{c^2}{2}-\frac{3}{4}\right) = 0.$$

Then we have

$$\frac{c^4}{4}+\frac{c^2}{2}-\frac{3}{4} = 0 \Leftrightarrow c^4+2c^2-3 = (c^2-1)(c^2+3) = 0,$$

so $c = 1$ (we exclude $c = -1$). Note that the function $f(x)$ is an odd function, so we can guess that this is the case. *Exercise.* Try to prove that if $f(x)$ is an increasing odd function and $\int_a^b f(x)dx = 0$, then $a + b = 0$.