1) Areas Between Curves

1. Find the area of the region enclosed by the curves $y = x^2$ and $y = x + 6$.

2. Determine the area between $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

2) Average Value of a Function

3. Compute the average value of the function $f(x) = 2x^3 - 3x^2 + x$ on the interval $[1, 4]$.

4. Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 9]$.

3) Volumes of Solids

5. Use the disk method to find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 0$ about the x-axis from $x = 0$ to $x = 2$.

6. Find the volume of the solid formed by rotating the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and $y = 0$ about the y-axis.

4) More Integration Problems

7. Find the area of the region bounded by $y = e^x$, $y = e^{-x}$, and $x = 1$.

8. Compute the volume of the solid obtained by rotating the region enclosed by $y = \ln(x)$, $y = 0$, and $x = e$ around the y-axis.

9. Determine the volume of the solid formed by rotating the region between the curves $y = x^3$ and $y = x$ around the line $y = x$.

Solutions to Worksheet on Applications of Integrals

November 3, 2023

Solutions

1) Areas Between Curves

1. The area of the region enclosed by $y = x^2$ and $y = x + 6$ is given by

$$\int_{-2}^{3} (x+6) - x^2\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3} = \frac{125}{6}.$$

2. Two curves intersect at $x = \pi/4$. The area between $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$ is

$$\int_0^{\frac{\pi}{4}} \cos(x) - \sin(x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x) dx = [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sqrt{2}.$$

2) Average Value of a Function

1. The average value of $f(x) = 2x^3 - 3x^2 + x$ on $[1, 4]$ is

$$\frac{1}{4-1}\left[\frac{x^4}{2} - x^3 + \frac{x^2}{2}\right]_1^4 = \frac{125}{6}.$$

2. The average value of $f(x) = \sqrt{x}$ on $[0, 9]$ is

$$\frac{1}{9}\left[\frac{2}{3}x^{3/2}\right]_0^9 = 2.$$

3) Volumes of Solids

1. The volume of the solid obtained by rotating $y = x^2$ about the x-axis from $x = 0$ to $x = 2$ is

$$\int_0^2 \pi (x^2)^2 dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}.$$

2. The volume of the solid formed by rotating $y = \sqrt{x}$, $0 \le x \le 4$, about the y-axis is

$$\int_0^2 \pi(4^2 - (y^2)^2) dy = \frac{128\pi}{5}$$

4) More Integration Problems

1. The area of the region bounded by $y = e^x$, $y = e^{-x}$, and $x = 1$ is

$$\int_0^1 e^x - e^{-x} dx = \left[e^x + e^{-x}\right]_0^1 = e + \frac{1}{e} - 2.$$

2. The volume of the solid obtained by rotating the region enclosed by $y = \ln(x)$, $y = 0$, and $x = e$ around the y-axis is

$$\int_0^1 \pi(e^2 - (e^y)^2) dy = \pi \left[e^2 y - \frac{e^{2y}}{2}\right]_0^1 = \frac{\pi(e^2 + 1)}{2}$$

3. The volume of the solid formed by rotating the region between the curves $y = x^3$ and $y = x$ around the line $y = x$ is

$$\int_0^1 \left(\frac{x - x^3}{\sqrt{2}}\right)^2 \cdot \sqrt{2} dx = \frac{\sqrt{2}}{2}\left[\frac{x^3}{3} - \frac{2}{5}x^5 + \frac{x^7}{7}\right]_0^1 = \frac{4\sqrt{2}}{105}$$