1. Find the derivative of $f(x) = x^x$.

2. Find all values a, if any, where the tangent line to $f(x) = \frac{x+1}{x-1}$ at a is parallel to the line $y = x - 1$.

3. Compute the following integral.

$$\int_0^{\sqrt{\pi}} x^3 \sin(-x^2)\,\mathrm{d}x$$

4. Find the volume of the solid obtained by rotating the region under the graph of $y = \frac{1}{x}$ for $x \geq 1$ about the x-axis.

5. Compute the following limits.
   (a) $\lim_{x\to-\infty} \frac{x(3x-4)+2}{5x^2-10}$
   (b) $\lim_{x\to-3} \frac{x^2-9}{x^2+2x-3}$

6. Diagonalize the following matrix.

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$$

1. Find the derivative of $f(x) = x^x$.

   We use logarithmic differentiation. Since

$$\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x)) = \frac{\mathrm{d}}{\mathrm{d}x}x\ln(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1,$$

   we have $f'(x) = f(x)(\ln(x) + 1) = x^x\ln(x) + x^x$.

2. Find all values a, if any, where the tangent line to $f(x) = \frac{x+1}{x-1}$ at a is parallel to the line $y = x - 1$.

   We have $f'(x) = \frac{x-1-(x+1)}{(x-1)^2} = \frac{-2}{x^2-2x+1}$ by the quotient rule. Since the line $y = x-1$ has a slope of 1, the tangent line at a is parallel to it if and only if $f'(a) = 1$. This implies $a^2 - 2a + 1 = -2$, i.e. $a^2 - 2a + 3 = 0$. Thus

$$a = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \frac{i\sqrt{8}}{2}.$$

   Hence no such real number a exists.

3. Compute the following integral.

$$\int_0^{\sqrt{\pi}} x^3 \sin(-x^2)\,\mathrm{d}x$$

   We first use integration by substitution. Let $u = -x^2$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$ and

$$\begin{aligned}\int_0^{\sqrt{\pi}} x^3\sin(-x^2)\,\mathrm{d}x &= \int_0^{-\pi}\frac{1}{2}u\sin(u)\,\mathrm{d}u\\ &= \left[-\frac{1}{2}u\cos(u)\right]_0^{-\pi} + \int_0^{-\pi}\frac{1}{2}\cos(u)\,\mathrm{d}u\\ &= \left[-\frac{1}{2}u\cos(u)\right]_0^{-\pi} + \left[\frac{1}{2}\sin(u)\right]_0^{-\pi}\\ &= -\frac{\pi}{2} + 0 = -\frac{\pi}{2}\end{aligned}$$

   using integration by parts.

4. Find the volume of the solid obtained by rotating the region under the graph of $y = \frac{1}{x}$ for $x \geq 1$ about the x-axis.

   This figure is called Torricelli's trumpet. It has infinite surface area but its volume is

$$\int_1^\infty \frac{\pi}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} -\frac{\pi}{x}\bigg|_1^t = \lim_{t\to\infty} \pi - \frac{\pi}{t} = \pi.$$

5. Compute the following limits.
   (a) $\lim_{x\to-\infty} \frac{x(3x-4)+2}{5x^2-10}$
   (b) $\lim_{x\to-3} \frac{x^2-9}{x^2+2x-3}$

   (a) $\frac{3}{5}$
   (b) $\frac{3}{2}$

6. Diagonalize the following matrix.

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$