MATH 10A PRACTICE EXAM 1

Please note that this practice test is designed to give you examples of the types of questions that will appear on the actual exam. Your performance on these sample questions should not be taken as an absolute prediction of your performance on the actual exam. When you take the actual exam, the questions you see will cover some of the same content included in the practice exam, as well as some content not tested on the practice exam.

(1) Find the inverse of the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ \boxed{2} & 1 & -3 & 0 & 1 & 0 \\ \boxed{1} & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; \begin{array}{l} 2 - 2 \times 1 \\ 3 - 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & \boxed{0} & 1 & 0 & 0 \\ 0 & -3 & \boxed{-3} & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; 2 + 3 \times 3$$

$$\left[\begin{array}{ccc|ccc} 1 & \boxed{2} & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; 3 + \frac{2}{3} \times 2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{2}{3} & 2 \\ 0 & -3 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; 2 \times \left(-\frac{1}{3}\right)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{2}{3} & 2 \\ 0 & 1 & 0 & \frac{5}{3} & -\frac{1}{3} & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} -\frac{7}{3} & \frac{2}{3} & 2 \\ \frac{5}{3} & -\frac{1}{3} & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

(2) (a) Find an equation of the sphere that passes through the point (6, -2, 3) and has center (-1, 2, 1).

(b) Find the projection of the $\mathbf{u} = [2, 4, 8]$ onto the vector $\mathbf{v} = [-1, 3, -5]$.

(c) Compute the following matrix product:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) Radius of the sphere = distance btw (6,-2,3) & (-1,2,1)

$$= \sqrt{(6-(-1))^2 + (-2-2)^2 + (3-1)^2} = \sqrt{69}$$

Center : $(-1, 2, 1)$

$\Rightarrow$ Equation : $(x+1)^2 + (y-2)^2 + (z-1)^2 = (\sqrt{69})^2 = 69$

(b) $\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$

$\vec{u} \cdot \vec{v} = 2(-1) + 4 \cdot 3 + 8 \cdot (-5)$
$= -30$

$\|\vec{v}\|^2 = (-1)^2 + 3^2 + (-5)^2 = 35$

$$= \frac{-30}{35} [-1, 3, -5] = \left[\frac{6}{7}, -\frac{18}{7}, \frac{30}{7} \right]$$

(c)

$$\begin{bmatrix} 1 \cdot 1 + 2 \cdot (-2) + 0 \cdot 0 & 1 \cdot 1 + 2 \cdot 0 + 0 \cdot 2 & 1 \cdot 0 + 2 \cdot (-3) + 0 \cdot 1 \\ 2 \cdot 1 + 0 \cdot 0 + (-3) \cdot 2 & 2 \cdot 1 + 0 \cdot 0 + (-3) \cdot 2 & 2 \cdot 0 + 0 \cdot (-3) + (-3) \cdot 1 \end{bmatrix}$$

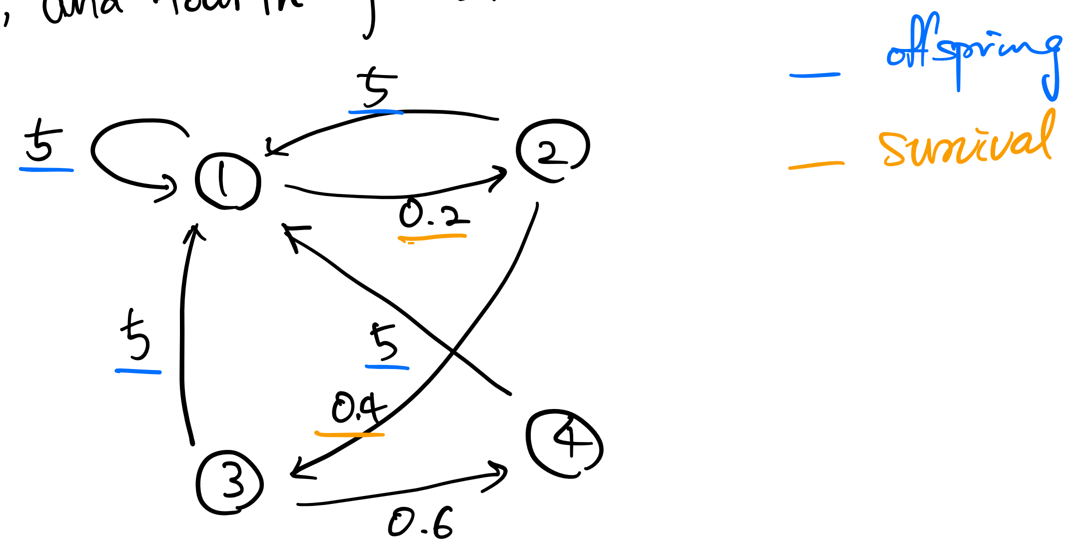
$$= \begin{bmatrix} -3 & 1 & -6 \\ 2 & -4 & -3 \end{bmatrix}$$

(3) Suppose that a species of bird lives for a maximum of four years and individuals of each age class produces five new offspring each year. In addition, the annual survival rates are 20 percent at age 1, 40 percent at age 2, and 60 percent at age 3.

(a) Draw the matrix diagram for this situation.

(b) Construct the matrix model from your diagram in part (a).

(a) Let ①, ②, ③, ④ for the first, second, third, and fourth years.



(b) Let $\vec{V}_t = \begin{bmatrix} x_t \\ y_t \\ z_t \\ w_t \end{bmatrix} = \begin{bmatrix} \text{first}_t \\ \text{second}_t \\ \text{third}_t \\ \text{fourth}_t \end{bmatrix}$

$$\Rightarrow \begin{aligned} x_{t+1} &= 5x_t + 5y_t + 5z_t + 5w_t \\ y_{t+1} &= 0.2x_t \\ z_{t+1} &= 0.4y_t \\ w_{t+1} &= 0.6z_t \end{aligned}$$

$$\Rightarrow \vec{V}_{t+1} = \begin{bmatrix} 5 & 5 & 5 & 5 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \vec{V}_t$$

(4) (a) Diagonalize the following matrix:

$$A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

(b) Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute $A^9\mathbf{v}$.

(a) $\det(A - \lambda I) = (-5-\lambda)(4-\lambda) - 2\cdot(-7)$

$= \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$

$\Rightarrow \lambda = 2, -3$

· $\lambda_1 = 2$. $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$\Rightarrow (A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -7x_1 + 2y_1 = 0$. Take $x_1 = 2, y_1 = 7$. $\vec{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

· $\lambda_2 = -3$, $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\Rightarrow (A - \lambda_2 I)\vec{v}_2 = \begin{bmatrix} -2 & 2 \\ -7 & 7 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_2 + 2y_2 = 0$. Take $x_2 = 1, y_2 = 1$. $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{2}{5} \end{bmatrix}$

(b) $A^9\vec{v} = \underbrace{\begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 2^9 & 0 \\ 0 & (-3)^9 \end{bmatrix}}\underbrace{\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$= \begin{bmatrix} 2^{10} & (-3)^9 \\ 7\cdot 2^9 & (-3)^9 \end{bmatrix}\begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{2^{10} - 3^{10}}{5} \\ \frac{7\cdot 2^9 - 3^{10}}{5} \end{bmatrix}$

(5) Consider the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$

(a) Using the initial condition $\mathbf{x}_0 = \cancel{[1,1]}\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ express the solution to the recursion $\mathbf{x}_{n+1} = B\mathbf{x}_n$ in terms of the eigenvalues and eigenvectors of B.

(b) Describe the long term behavior of $\mathbf{x}_n$.

(a) Let's diagonalize B.

$$\det(B-\lambda I) = (1-\lambda)\left(\tfrac{1}{2}-\lambda\right) = 0 \Rightarrow \lambda = 1, \tfrac{1}{2}.$$

- $\lambda_1 = 1$, $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \Rightarrow (A - \lambda_1 I)\vec{v_1} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $\Rightarrow y_1 = 0$. Take $x_1 = 1$, $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $\lambda_2 = \frac{1}{2}$, $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow (A - \lambda_2 I)\vec{v_2} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  $\Rightarrow \frac{1}{2}x_2 + y_2 = 0$. Take $x_2 = 2$, $y_2 = -1$. $\vec{v_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\Rightarrow B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\vec{X_n} = B^n \vec{X_0} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^n \end{bmatrix}}\ \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & \frac{1}{2^{n-1}} \\ 0 & -\frac{1}{2^n} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 - \frac{1}{2^{n-1}} \\ \frac{1}{2^n} \end{bmatrix}$$

(b). As $n \longrightarrow \infty$, $\frac{1}{2^n} \longrightarrow 0$, so

$$\vec{X_n} = \begin{bmatrix} 3 - \frac{1}{2^{n-1}} \\ \frac{1}{2^n} \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$