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MATH 10A PRACTICE EXAM 2

Please note that this practice test is designed to give you examples of the types of questions that will appear on the actual exam. Your performance on these sample questions should not be taken as an absolute prediction of your performance on the actual exam. When you take the actual exam, the questions you see will cover some of the same content included in the practice exam, as well as some content not tested on the practice exam.

(1) (a) Find the derivative of $f(x) = \ln(\ln(\ln x))$.
(b) Find the derivative of $f(x) = x^2 e^{8x} \sin(x^2)$.

(a) By chain rule,

$$f'(x) = \frac{1}{\ln(\ln(x))} \cdot \ln(\ln(x))' = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \ln(x)'$$

$$= \boxed{\frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}.}$$

(b) By product rule & chain rule,

$$f'(x) = (x^2 e^{8x})' \cdot \sin(x^2) + x^2 e^{8x} \cdot \sin(x^2)'$$

$$= (2x e^{8x} + x^2 \cdot 8e^{8x}) \sin(x^2) + x^2 e^{8x} \cdot 2x \cos(x^2)$$

$$= \boxed{(2x e^{8x} + 8x^2 e^{8x}) \sin(x^2) + 2x^3 e^{8x} \cos(x^2)}$$

(2) Sketch the graph of $f(x) = \frac{(x+1)^2}{x^2+1}$. Find and label all local and global maximum and minimum points and inflection points.

First, we have $f(x) = \dfrac{x^2+2x+1}{x^2+1} = 1 + \dfrac{2x}{x^2+1}$. This function is defined for all real x.

Critical points : $f'(x) = \dfrac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \dfrac{-2x^2+2}{(x^2+1)^2} = -2 \cdot \dfrac{(x+1)(x-1)}{(x^2+1)^2} = 0$

$\Rightarrow x = -1, 1$. & $f(-1) = 0$, $f(1) = 2$.

Now, observe sign of $f'(x)$. Since $(x^2+1)^2$ is always positive, sign of $f'(x)$ is the same as that of $-2(x+1)(x-1)$.

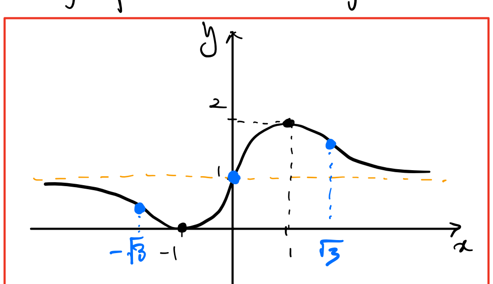
	$x<-1$	$-1<x<1$	$x>1$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

$1 + \frac{2\sqrt{3}}{4} =$

Now, let's investigate limits of $f(x)$ as $x \to \pm\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 + \frac{2x}{x^2+1} = \lim_{x\to\infty} 1 + \frac{2}{x+\frac{1}{x}} = 1$$

and similarly, $\lim_{x\to-\infty} f(x) = 1$. Hence $y=1$ is a horizontal asymptote of $y = f(x)$.



From graph, the critical points $\boxed{(-1, 0)}$ & $\boxed{(1, 2)}$ are not only local min/max, but also absolute min/max. (0 & 2 respectively)

For reflection points, we observe sign of $f''(x)$.

$$f''(x) = \frac{(-4x)(x^2+1)^2 - (-2x^2+2)\cdot 2\cdot(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{-4x(x^2+1) - 4x(-2x^2+2)}{(x^2+1)^3} = \frac{4x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$$

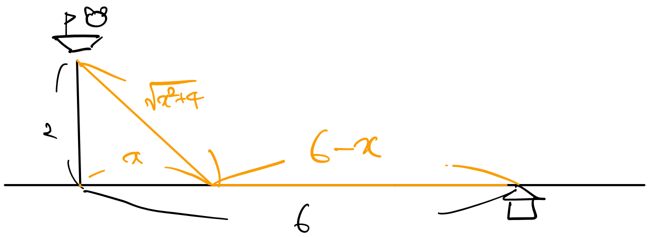
	$x<-\sqrt{3}$	$-\sqrt{3}<x<0$	$0<x<\sqrt{3}$	$x>\sqrt{3}$
$f''(x)$	$-$	$+$	$-$	$+$

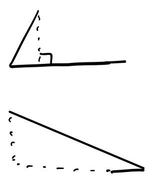
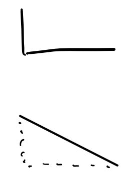
The left table tell us convexity of graph, and also

$$\boxed{\begin{array}{l} (-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, 1-\frac{\sqrt{3}}{2}) \\ (0, f(0)) = (0, 1) \\ (\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, 1+\frac{\sqrt{3}}{2}) \end{array}}$$

are inflection points.

(3) Oski is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down a straight shoreline from the point nearest the boat. He can row 2 miles per hour and can walk 5 miles per hour. Where should Oski land the boat to reach the village in the least amount of time?



First, note that we only need to consider $0 \le x \le 6$ in the above picture. (Going in  takes longer than )

Time $T(x) = \dfrac{\sqrt{x^2+4}}{2} + \dfrac{6-x}{5}$. Minimize $T(x)$.

$$T'(x) = \frac{1}{2}\cdot\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\cdot 2x - \frac{1}{5} = \frac{x}{2\sqrt{x^2+4}} - \frac{1}{5}.$$

Critical number : $\dfrac{x}{2\sqrt{x^2+4}} - \dfrac{1}{5} = 0$, $\dfrac{x}{2\sqrt{x^2+4}} = \dfrac{1}{5}$

$$\Rightarrow 25x^2 = 4(x^2+4), \quad x = \frac{4}{\sqrt{21}} \quad (\text{we have } x \ge 0).$$

$$T''(x) = \frac{2\sqrt{x^2+4} - x\cdot 2\cdot\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\cdot 2x}{4(x^2+4)} = \frac{(x^2+4) - x^2}{2(x^2+4)^{3/2}} = \frac{2}{(x^2+4)^{3/2}} > 0$$

$\Rightarrow T(x)$ is concave upward for all x, so $x = \dfrac{4}{\sqrt{21}}$ gives local & absolute minimum.

$$T\left(\frac{4}{\sqrt{21}}\right) = \frac{1}{2}\cdot\sqrt{\frac{16}{21}+4} + \frac{6-\frac{4}{\sqrt{21}}}{5} = \frac{5}{\sqrt{21}} + \frac{6}{5} - \frac{4}{5\sqrt{21}}$$

$$= \boxed{\frac{\sqrt{21}+6}{5}}$$

(4) Let $f(x) = \ln(x)$.

(a) Find the degree four Taylor polynomial of $f(x) = \ln(x)$ centered at $x = 1$.

(b) Use the degree four Taylor polynomial to approximate $\ln(1.1)$.

(a) $T_4 f(x) = \sum_{k=0}^{4} \frac{f^{(k)}(1)}{k!}(x-1)^k$.

$f'(x) = \frac{1}{x}$, $f'(1) = 1$ $\qquad f(1) = 0$

$f''(x) = -\frac{1}{x^2}$, $f''(1) = -1$

$f^{(3)}(x) = \frac{2}{x^3}$, $f^{(3)}(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4}$, $f^{(4)}(1) = -6$.

$$\Rightarrow T_4 f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{6}(x-1)^3 + \frac{f^{(4)}(1)}{24}(x-1)^4$$

$$= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

(b) $\ln(1.1) \approx T_4 f(1.1) = \boxed{0.1 - \frac{1}{2}0.1^2 + \frac{1}{3}0.1^3 - \frac{1}{4}0.1^4}$

(5) Consider the equation $y^2 = e^{x^2} + 2x$.

(a) Find dy/dx.

(b) Find d^2y/dx^2.

(c) Find the equation of the tangent line at (0,-1).

(a) By implicit differentiation,

$$2y\frac{dy}{dx} = 2xe^{x^2} + 2, \quad \frac{dy}{dx} = \boxed{\frac{xe^{x^2}+1}{y}}$$

(b) By implicit differentiation again, applied to $\frac{dy}{dx} = \frac{xe^{x^2}+1}{y}$:

$$\frac{d^2y}{dx^2} = \frac{(e^{x^2} + x\cdot 2xe^{x^2})y - (xe^{x^2}+1)\frac{dy}{dx}}{y^2} = \frac{(1+2x^2)e^{x^2}y - (xe^{x^2}+1)\cdot\frac{xe^{x^2}+1}{y}}{y^2}$$

$$= \frac{(1+2x^2)e^{x^2}y^2 - (xe^{x^2}+1)^2}{y\cdot y^2} = \frac{(1+2x^2)e^{x^2}(e^{x^2}+2x) - (xe^{x^2}+1)^2}{y\cdot(e^{x^2}+2x)}$$

$$= \frac{2x^2e^{2x^2} + e^{2x^2} + 4x^3e^{x^2} + 2xe^{x^2} - x^2e^{2x^2} - 2xe^{x^2} - 1}{y(e^{x^2}+2x)}$$

$$= \boxed{\frac{(x^2+1)e^{2x^2} + 4x^3e^{x^2} - 1}{y(e^{x^2}+2x)}}$$

(c) Slope : $\left.\frac{dy}{dx}\right|_{(x,y)=(0,-1)} = \frac{0\cdot e^{0^2}+1}{-1} = -1$

$$\Rightarrow \boxed{y = -1\cdot(x-0) + (-1) = -x - 1.}$$