Week 10, November 1

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- Theorems that you need to know:
  - 8.1, 8.2, 8.3: null(T^k) "increases" as k increase, but stabilizes at some point. If null(T^m) = null(T^{m+1}), then the following spaces are all same (null(T^m) = null(T^{m+1}) = null(T^{m+2}) = null(T^{m+3}) = $\cdots$). 8.3 tells us that $m \leq \dim V$. Exercise 8A.6, 8A.7, and 8A.8 are "dual" statements of these theorems on range(T^k) instead of null(T^k).
  - 8.11: intersection of generalized eigenspaces for different eigenvalues are always $\{0\}$.
  - 8.17: nilpotent $\Rightarrow$ all eigenvalues are 0. For $F = \mathbb{C}$, converse holds.
  - 8.18: minimal polynomial of nilpotent operator and *strictly* upper-triangular matrices.
  - 8.22, especially (c): for *any* $T : V \to V$ defined on a *complex* vector space V, we have a decomposition of V into a direct sum of generalized eigenspaces. Almost equvalent to 8.9.
- Combining 8.18 and 8.22 tells you that there exists a basis where the corresponding matrix representation is a block-diagonal form, where each block is upper triangular with (same) eigenvalues on the diagonals.
- (8A.1) Direct consequence of 8.2.
- (8A.6, 8A.7) These are "dual theorems" of 8.1 and 8.2.
- (8A.13) $(TS)^k = TSTSTS \cdots TS = T(STSTS \cdots T)S = T(ST)^{k-1}S$.
- (8A.14) In this sense, you can think nilpotent matrices as the matrices that are "the most non-diagonalizable" matrices.
- (8A.24) Here's a computation for (b), where

$$T = \begin{bmatrix} -3 & 9 & 0 \\ -7 & 9 & 6 \\ 4 & 0 & -6 \end{bmatrix}.$$

We'll construct a basis $\beta = (v_1, v_2, v_3)$ that gives a *Jordan form* of T, not just upper-triangular (so better than the solution). Note that this is guaranteed since T is nilpotent and its minimal polynomial is $p(z) = z^3$, whose zeros are all in F for both $F = \mathbb{R}$ and $F = \mathbb{C}$. First of all, once we write it as an upper-triangular matrix, the diagonal entries are all zero since 0 is the only eigenvalue of T. Hence it should have a form of

$$[T]_\beta = \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

Equivalently, the vectors v_1, v_2, v_3 satisfy

$$\begin{cases} Tv_1 = 0 \\ Tv_2 = (*)v_1 \\ Tv_3 = (*)v_2. \end{cases}$$

where $*$ could be 0 or 1 - we need to decide which *one* is the correct *one* (yes, the answer would be both 1). First, $v_1 \in \operatorname{null}(T)$, and one can check that the null space has dimension 1, spanned by

$$v_1 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}.$$

Now, we have two possibilities, $Tv_2 = 0$ or $Tv_2 = v_1$. The correct answer is $Tv_2 = v_1$, since the dimension of the null space is 1 and we cannot choose another v_2 which is linearly independent with v_1. You can find that the equation is solvable, and one of the solution is

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

In fact, there are infinitely many solutions, but all of them give a vector linearly independent with v_1 (why?). By the same argument, we should have $Tv_3 = v_2$ and one possible solution is One such solution is

$$v_3 = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{2}{9} \end{bmatrix}.$$

For (a), there is an easy solution (hint: we discussed it last week), and for (c), the "standard" choice already works.

- (8B.3) As a side note, two operators T_1 and T_2 are called *conjugate/similar* if there exists an invertible linear map S such that $T_2 = S^{-1}T_1S$. Two linear operators are conjugate if and only if they admit the same matrix representation with a suitable choice of basis, i.e. $[T_1]_{\beta_1} = [T_2]_{\beta_2}$ (try to prove this). In other words, T_1 and T_2 are *essentially same linear operators*. This is an extremely important concept, but unfortunately it is not emphasized in our textbook. Conjugate linear maps share a lot of features, including eigenvalues, minimal polynomial, characteristic polynomial, etc. The theory of Jordan form tells you that **Jordan form completely determines a linear map up to conjugation**.

- (8B.11, 8B.12, 8B.13, 8B.14, 8B.21) The easiest way to construct such T is via Jordan block. 8B.21 is a generalization of the first four exercises. Key fact is the following: let

$$J_{\lambda,n} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

be the $n \times n$ Jordan block with eigenvalue λ. Then its minimal polynomial is $p(z) = (z - \lambda)^n$. Now, when $F = \mathbb{C}$, we can always find a basis β with

$$[T]_\beta = \begin{bmatrix} J_{\lambda_1,n_1} & 0 & 0 & \cdots & 0 \\ 0 & J_{\lambda_2,n_2} & 0 & \cdots & 0 \\ 0 & 0 & J_{\lambda_3,n_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{\lambda_k,n_k} \end{bmatrix}$$

where λ_i's are not necessarily distinct. Then the characteristic polynomial of T is

$$\mathrm{char}_T(z) = \prod_{i=1}^{k} (z - \lambda_i)^{n_i}$$

(again, some of λ_i's could be the same), and the minimal polynomial is

$$p_T(z) = \prod_{\lambda} (z - \lambda)^{n_\lambda}, \quad n_\lambda = \begin{cases} \max_i\{n_i : \lambda_i = \lambda\} & \lambda_i = \lambda \text{ for some } i \\ 0 & \lambda_i \neq \lambda \forall i \end{cases}$$

For example, you can take

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

for 8B.13, and

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

for 8B.14.

- (8B.15, 8B.16) These are nilpotent operators (why?), so you only need to take power until you get a zero operator. Now, try to find basis whose correponding matrix representation is a Jordan form, and compute minimal polynomials from it.

1 Problems

- Recommended problems: 8A.3, 8A.16
- Additional problems:

1. Why Theorem 8.17 (b) is not true for $F = \mathbb{R}$?
2. (Continue from 8A.13) Let $S, T \in \mathcal{L}(V)$ be nilpotent operators. Is ST also nilpotent?
3. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map defined as $Te_i = e_1 + e_2 + \cdots + e_i$ for $1 \leq i \leq n$. Find eigenvalues of T and a basis β where $[T]_\beta$ is a Jordan form.
4. Prove that the following matrices have the same eigenvalues, minimal polynomials and characteristic polynomial.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$