Week 11, November 8

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- I was a bit ahead last week and accidentally gave a preview about Jordan forms. This week, I gave more examples and some comments on a proof of the main theorem (8.45, 8.46). Chapter 9 will be explained in detail next week. In summary, the main goal of Chapter 9 is to define *determinant*.
- Theorems that you need to know:
  - 8.45, 8.46: Existence of a Jordan basis. Note that we are considering linear operators over $\mathbb{C}$; over $\mathbb{R}$, we may not have eigenvalues.
    * There *is* a theory of Jordan form over $\mathbb{R}$, but it is out of scope of this class.
  - 9.4: Like linear maps, bilinear maps are also determined by their values on pair of basis vectors $\beta(v_i, v_j)$. We can write it as a matrix, which is a matrix representation of a bilinear map.
  - 9.7: Change of basis for bilinear maps.
- For the existence of Jordan basis, it reduces to a statement on a nilpotent operator (by considering $T - \lambda I$ instead of T). The main idea of the proof in 8.45 is to decompose V into several T-invariant subspaces, each of them corresponds to a single Jordan block.
- Here's another example of finding Jordan form. Let

  $$T = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

  This is a special kind of matrix that you've learned - companion matrix (Exercise 5B.16).

  Especially, we can immediately find the minimal polynomial from the last column: $p(z) = z^3 + 3z^2 - 4$. This factors as $p(z) = (z+2)^2(z-1)$, hence T has two eigenvalues $\lambda = -2, 1$. For $\lambda = 1$, solving $(T - I)v = 0$ gives an eigenvector

$$v_1 = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}.$$

Can we go further? Are there any v_2 such that $(T + 2I)v_2 = v_1$? We get an equation

$$\begin{cases} -x + 4z = 4 \\ x - y = 4 \\ y - 4z = 1 \end{cases}$$

but this is not solvable: you can derive a contradiction, or by observing that the sum of the left hand sides are 0, while the sum of the right hand sides is 9.

For $\lambda = -2$, solving $(T + 2I)v = 0$ shows that the *eigenspace* $E(-2, T)$ has dimension 1, spanned by

$$v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

From $\dim E(1, T) + \dim E(-2, T) = 1 + 1 = 2 < 3$, we can conclude that T is not diagonalizable, and we need a generalized eigenvector for $\lambda = -2$. Solving $(T + 2I)v = v_1$ gives

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(Again, there are infinitely many choices other than above, giving the same Jordan form.) We can't go further; there is no v_4 such that $(T + 2I)v_4 = v_3$ (why?).

Now, with respect to the basis $\beta = (v_1, v_2, v_3)$, we have

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

For each eigenvalue $\lambda = 1$ and -2, we have Jordan blocks of size 1 and 2, respectively. In other words, we have

$$T = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1}.$$

Is it possible to guess the Jordan form without computing all the generalized eigenvectors (and Jordan basis)? In this example, the answer is yes, from the minimal polynomial. The power of the factor $(z + 2)$ is two, meaning that there should be a Jordan block of size two for $\lambda = -2$. Then there's only one room for $\lambda = 1$, which means that the corresponding block has size one (and there's only one block).

Let's do one more example. Here is another companion matrix:

$$T = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}.$$

The minimal polynomial is $p(z) = (z+1)^3$, hence the only eigenvalue is $\lambda = -1$. One can find that the null space of

$$T + I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}.$$

is 1, which is spanned by

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

By applying the same technique above, we can find v_2, v_3 satisfying $(T+I)v_2 = v_1$ and $(T + I)v_3 = v_2$; one can take

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

You can't go further, and (v_1, v_2, v_3) is a Jordan basis. Again, you can guess the Jordan form only using the minimal polynomial. See also the additional problems 4 and 5.

- (8C.5) It is easy to find minimal polynomial of a linear operator on a 2-dimensional vector space (we have a formula for it! Exercise 5B.11). See also the additional problem 1.

- (8C.9, 8C.13) Jordan form of nilpotent operators.

- Chapter 9: You have already learned a bilinear form in Math 54: the inner product of two vectors in $\mathbb{R}^n$. Bilinear form is a generalization of it. It is a map $B : V \times V \to F$ which is linear on each argument. As linear maps, bilinear maps are also determined by their values on basis vectors. Especially, for a given basis $\beta = (v_1, \ldots, v_n)$, we get a matrix representation of B (write as $[B]_\beta$), whose entries are $B(v_i, v_j)$. We have a change of basis formula (Theorem 9.7).

  We have two types of them: *symmetric* and *alternating*. Symmetric bilinear maps are generalized inner products, which are highly related to the concept of orthogonality (Chapter 6 and 7 - that's why we are doing Chapter 9 first). Especially, you can do everything that you have done for inner products, e.g. Gram–Schimidt process.

  Alternating bilinear maps could be less familiar, which are the maps satisfying $B(y, x) = -B(x, y)$. Later we define m-bilinear maps, and the main result is that there's essentially only one n-bilinear maps on a vector space of dimension n, which is called *determinant* (Theorem 9.37). The main goal of Chapter 9 is to define *determinant* in the best way using alternating forms, which would recover the determinant formula of 2 by 2 and 3 by 3 matrices you saw before.

1 Problems

- Recommended problems: 8C.5 (or more generally, practice how to find Jordan form/basis). 8C.6 (try with general $\mathcal{P}_n(\mathbb{R})$, not just $n = 4$), 8C.11, 8C.13, 9A.8, 9A.9

- Additional problems:

  1. For the above 3 by 3 examples of Jordan forms, find T^{10}.

  2. Find a Jordan basis of $T : \mathcal{L}(\mathbb{C}^5) \to \mathcal{L}(\mathbb{C}^5)$, defined as

  $$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

  3. Let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathcal{L}(\mathbb{C}^2)$. Find the precise condition when T is not diagonalizable, in terms of a, b, c, d. Check your answer with the Exercise 8C.5.

Hint: Exercise 5B.11. [1]

4. Let $T \in \mathcal{L}(\mathbb{C}^n)$ be a linear map that can be represented as a companion matrix. Show that all the eigenspaces have dimension 1. Hence, each eigenvalue has single Jordan blocks.

5. Let V be a complex vector space of (complex) dimension ≤ 3. Let $S, T \in \mathcal{L}(V)$. Prove that if S, T have the same minimal polynomial, then they have the same Jordan form. In general, this is not true for $\dim V \geq 4$; you can find a counterexample in the previous week's note.

6. (Optional) If you want something more interesting, read page 319 and 320 and try 8C.2. If you are brave enough, think about how to find *cube root* of $I + T$ (i.e. $S \in \mathcal{L}(V)$ such that $S^3 = I + T$) when T is nilpotent. If you want more efficient way to do this, search about *generalized binomial theorem* or *Newton's binomial theorem*.

[1]Answer: $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$ and $(a+d)^2 - (ad - bc) = 0$.