Week 12, November 15

Seewoo Lee

- Make sure that you do the extra HW problem on a Jordan form.
- Theorems that you need to know:
  - 9.28, 9.29: n-dimensional vector space cannot possess nonzero m-alternating form when $m > n$.
  - 9.30: This is *why* it is called alternating. Note that some of the textbooks use this as a definition, but our book's definition is 9.27 and we take this one. There is a mathematical world where 9.27 and 9.30 are not equivalent.
  - 9.34, 9.35: Properties of permutation, swapping two of them flips a sign.
  - 9.37: Main theorem of chapter 9, and the way to define determinant.
  - 9.41: Definition of a determinant, $\alpha_T = (\det T)\alpha$.
  - 9.36, 9.46: Formula of a determinant, in terms of signs of permutations and entries.
  - 9.49, 9.50, 9.51, 9.52: The later three follows from 9.49, multiplicativity of determinant. 9.52 is important. It tells you that determinant of a linear map equals to the determinant of a matrix representation, no matter which basis you choose.
  - (9.62), 9.63, 9.64: Charateristic polynomial is $\det(zI - T)$.
  - 9.65: Cayley–Hamilton theorem.
- I explained the definition of m-(alternating) linear maps and the dimension formula $\dim V_{\mathrm{alt}}^{(m)} = \binom{n}{m}$. Especially, there's no nonzero m-alternating map when $m > n$. The key fact is when $m = n$, $\dim V_{\mathrm{alt}}^{(n)} = 1$ (Theorem 9.37), and we use this to define the determinant of a linear map (Definition 9.41). I explained how to recover the formula $ad - bc$ using this definition. Here's a good exercise for you: **recover the determinant formula for 3 by 3 matrices using the definition.** Out of $3^3 = 27$ terms, only $3! = 6$ of them are possibly nonzero, and you need to determine the signs.

- General formula: there's a dirty-looking formula for a determinant of a n by n matrix (Theorem 9.46):

$$\det A = \sum_{(j_1,\ldots,j_n)\in \mathrm{perm}(n)} \mathrm{sign}(j_1,\ldots,j_n) A_{j_1,1}A_{j_2,2}\cdots A_{j_n,n}.$$

  Proof is essentially the same as $n = 2$ case. Note that the textbook defines $\mathrm{sign}(j_1,\ldots,j_n)$ in a slightly different way (9.32), but it coincides with the definition I explained: $(-1)^{\text{number of swaps to order it}}$[1].

  To prove some properties of determinants, I recommend to use the original definition $\alpha_T = (\det T)\alpha$ instead of the above explicit formula. For example, it is better to use this to do Exercise 9C.5, where the book's solution uses the explicit formula (and hard to follow!).

- There are several nice properties of the determinant, and one of the most important one is multiplicativity: $\det(ST) = \det(S)\det(T)$ (Theorem 9.50). **Try to prove Theorem 9.50, 9.51, 9.52 yourself without looking at the proof, using Theorem 9.49**.

1 Problems

- Recommended problems: 9B.2, 9C.2, 9C.6, 9C.13
- Additional problems:

  1. Prove Theorem 9.48 (determinant of a upper triangular matrix) without using the explicit formula, but only using the definition 9.41.
  2. Explain why the determinant of a lower triangular matrix is equal to the product of the diagonal entries.
  3. Let $T : V \to V$ be a linear map and β_1, β_2 be two basis of V. Explain why $\det([T]_{\beta_1}) = \det([T]_{\beta_2})$. This gives an alternative way to define a determinant of a linear map.
  4. (Not easy!! If you know how to do this, let me know and I can give you a reward) Let V be a real vector space and $S, T \in \mathcal{L}(V)$. Prove that $\det(S^2 + T^2) \geq 0$.

[1] The book's definition is better in the sense that we don't have any ambiguity with the definition. For the definition using the number of swaps, we need to prove that the result does not depends on the history of swaps, which is a nontrivial fact.