Week 16, December 13

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1 Comments on the mock final

There are not complete answers to the questions, but rather comments and additional exercises that you can try. Especially, you may need to write more formally in the actual exam.

1. First of all, A is a hermitian matrix, i.e. $\overline{A}^{\intercal} = A^* = A$. In other words, A defines a self-adjoint operator. In that case, it is guaranteed that the eigenvectors for different eigenvalues are orthogonal (try to prove this if you haven't, you can find the answer here which uses Problem 5 below). The eigenvalues are 0 and 3, so they are distinct and the corresponding eigenvectors are automatically orthogonal (but you still need to normalize). Note that (standard) inner product on $\mathbb{C}^2$ is defined as

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1\overline{x_2} + y_1\overline{y_2},$$

where there is a complex conjugation in the second vector (if we don't include this, then it *does not* define an inner product.

- Assume that T is a normal operator and v, w are eigenvectors of two different eigenvalues λ, μ. Show that v, w are orthogonal.
- On $\mathbb{C}^2$, check that the *naive* inner product

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := x_1y_1 + x_2y_2$$

does not define an inner product. Which properties fail?

2. This Exercise 5B.3, but also asking for eigenvalues and minimal polynomials. Before you learned about Jordan forms, you can find the minimal

polynomial by 1) knowing that there are two eigenvalues $\lambda = 0, n$, so the minimal polynomial would have a form of $p(z) = z^a(z-n)^b$ with $a, b \geq 1$, and 2) check that $T(T - nI) = 0$, so the minimal possible a, b ($a = b = 1$) actually gives the minimal polynomial (you don't need to go higher exponent). For now, you learned eigenvalues/vectors, and one can check that $\dim E(0, T) = n - 1$ and $\dim E(n, T) = 1$, so $\dim E(0, T) + \dim E(n, T) = n$ and T is diagonalizable. This gives you an another way to find a minimal polynomial.

- Check that A is self-adjoint, so is normal. By the theory, eigenvectors for distinct eigenvalues should be orthogonal: check this, too.

3. Remind that there are several things that you need to check for being an inner product. The answer could be simpler than you think: since the interval is symmetric, any odd function integrates to zero. Now choose the first vector as a constant polynomial, and the second one as a odd polynomial $p(t)$ so that $\langle 1, p(t)\rangle = 0$, where you can simply choose as $p(t)$. Of course, you need to normalize these with respect to the norm $\|f(t)\| = \left(\int_{-2}^{2} f(t)^2 \mathrm{d}t\right)^{1/2}$.

   - Try to do the same question with

   $$\langle f(t), g(t)\rangle = \int_{-2}^{2} f(t)g(t)(t^2+1)\mathrm{d}t.$$

   In fact, you can replace $t^2 + 1$ with any other nonnegative nonzero continuous function $w(t)$ on $[-2, 2]$.

4. Finding Jordan form is explained a lot, so let me skip the details. The matrix Q is nothing but the basis of (generalized) eigenvectors. More precisely, if you write your Jordan form as

   $$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

   then the corrsponding Q would be

   $$Q = \begin{pmatrix} v_1 & v_2 & w_2 \end{pmatrix}$$

   where v_1, v_2 are genuine eigenvectors of $\lambda = 1, 2$ respectively, and w_2 is a generalized eigenvector for $\lambda = 2$ (i.e. $(T - 2I)w_2 = v_2$).

- Check that this actually satisfy the desired relation $A = QJQ^{-1} \Leftrightarrow AQ = QJ$.
- What happens if the characteristic polynomial is $(\lambda - 2)^3$? Describe all the possible scenario.

5. Using $Tv = \lambda v$, you need to prove $T^*v = \overline{\lambda}v$. One way is to considering $\|T^*v - \overline{\lambda}v\|^2$ and prove that it is zero, by writing it as (self-)inner product and use normality. Especially, we have $\|Tw\| = \|T^*w\|$ for normal T and any $w \in V$ (Theorem 7.20), and this will be useful.

   More direct way to observe the following: if $(T - \lambda I)^* = T^* - \overline{\lambda}I$, and $T - \lambda I$ is normal if T is. Then you can apply the theorem to the operator $T - \lambda I$.

   - Prove Theorem 7.20 yourself if you haven't done it.
   - For a normal operator $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{C}$, prove that $(T - \lambda I)^* = T^* - \overline{\lambda}I$ and $T - \lambda I$ is also a normal operator.
   - Find an operator T which is not a normal operator and fails to satisfy the conclusion of the problem.

6. This is a "dual" theorem of Theorem 8.2 of the book. Both can be proven in a similar way, but also these two theorems (for null and range) are equivalent by the fundamental theorem of linear algebra.

   - Assume that $\mathrm{null}(T^m) = \mathrm{null}(T^{m+1})$ for some m. Prove that $\mathrm{range}(T^m) = \mathrm{range}(T^{m+1})$.

7. If T is diagonalizable, it means that $G(\lambda, T) = E(\lambda, T)$ for all λ. In other words, any vector v satisfying $(T - \lambda I)^N v = 0$ for some $N \geq 1$ must satisfy $(T - \lambda I)v = 0$, which proves the equality of null spaces.

   For the other direction, assume T is not diagonalizable. We still have a Jordan form, hence we have a *generalized* eigenvector $w \in V$ where $v = (T - \lambda I)w$ is a genuine eigenvector (consider the "second" vector in a chain). Then $(T - \lambda I)w = v \neq 0$ but $(T - \lambda I)^2 w = (T - \lambda I)v = 0$, so $w \in \mathrm{null}((T - \lambda I)^2) - \mathrm{null}(T - \lambda I)$.

8. There are basically two proofs that you usually see, one in the textbook and one uses "dual basis". Since the textbook proof can be found in the textbook, let me explain more about the "dual basis" proof.

   For $m = 1$, observe that $V^{(1)} = V'$, the dual space, and the statement reduces to $\dim(V') = \dim(V)$, and we proved this by constructing a dual basis: for a basis $\beta = (v_1, \dots, v_n)$ of V, the dual basis $\beta' = (\varphi_1, \dots, \varphi_n)$ defined as $\varphi_i(v_j) = \delta_{ij}$ is indeed a basis, where you need to prove linear independence and spanning property.

For general m, the "dual basis" will be the following. For each m-tuple $(i_1, \dots, i_m)$ of integers from 1 to n, define $\varphi_{i_1,\dots,i_m}$ as

$$\varphi_{i_1,\dots,i_m}(v_{j_1}, \dots, v_{j_m}) = \begin{cases} 1 & (j_1, \dots, j_m) = (i_1, \dots, i_m) \\ 0 & \text{otherwise} \end{cases}.$$

and *linearly extend* to $V \times \cdots \times V$. One thing to note is that any input $(w_1, \dots, w_m)$ can be written in terms of the basis vectors in β, and m-linearity uniquely determines $\varphi_{i_1,\dots,i_m}(w_1, \dots, w_m)$ as

$$\begin{aligned} \varphi_{i_1,\dots,i_m}(w_1, \dots, w_m) &= \varphi_{i_1,\dots,i_m}\left(\sum_{k=1}^{n} a_{1,k}v_k, \dots, \sum_{k=1}^{n} a_{m,k}v_k\right) \\ &= \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} a_{1,k_1} \cdots a_{m,k_m} \varphi_{i_1,\dots,i_m}(v_{k_1}, \dots, v_{k_m}) \\ &= a_{1,i_1} \cdots a_{m,i_m}. \end{aligned}$$

Now, goal is to prove the same thing: the vectors $(\varphi_{i_1,\dots,i_m})_{1 \le i_1,\dots,i_m \le n}$ are linearly independent and span $V^{(m)}$. Once we know this, the dimension is equal to the number of m-tuples $(i_1, \dots, i_m)$, which is n^m. To prove linear independence, assume

$$\sum_{k_1,\dots,k_m} c_{k_1,\dots,k_m} \varphi_{k_1,\dots,k_m} = 0$$

i.e. it is a zero m-linear map. Then we get a zero vector for any input, especially for $(v_{i_1}, \dots, v_{i_m})$, the equation reduces to $c_{i_1,\dots,i_m} = 0$. Since this is true for any tuples, we get all the coefficients to be zero and $\varphi_{i_1,\dots,i_m}$ are linearly independent. Proof for the spanning part is similar to $m = 1$ case: you will eventually show that for any $\varphi \in V^{(m)}$,

$$\varphi = \sum_{k_1,\dots,k_m} \varphi(v_{k_1}, \dots, v_{k_m}) \varphi_{k_1,\dots,k_m}.$$

- Complete the proof.
- Let $f, g \in V^{(m)}$ and assume that $f(v_{i_1}, \dots, v_{i_m}) = g(v_{i_1}, \dots, v_{i_m})$ for any $(i_1, \dots, i_m)$. Prove $f = g$.