Week 3, September 13

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1 Discussion notes

- Intuitively but not mathematically, *basis* is the "right amount of vectors" that is 1) not too much so that they are linearly independent, and 2) not too few so that they spans the whole space. The "right amount" is called *dimension* of a vector space. Especially, the dimension does not depend on the choice of a basis (Theorem 2.34, although it is defined in terms of a basis). If there are too many of them, you can always remove some of them to make it as a basis (Theorem 2.30). If there are too few vectors, we can always add more (Theorem 2.32). Especially, Theorem 2.32 also proves that any subspace U of V has a complement W: $V = U \oplus W$ (Theorem 2.33). However, if you have n vectors in a n-dimensional vector space, they might not form a vector space. For example, two linearly *dependent* vectors $(1, 2)$ and $(2, 4)$ in $\mathbb{R}^2$ do not form a basis.

  We consider the example $\mathcal{P}_3(\mathbb{R})$, the space of degree ≤ 3 polynomials with coefficients in $\mathbb{R}$. We have a "easy" basis of it: $\{x^3, x^2, x, 1\}$. They span $\mathcal{P}_3(\mathbb{R})$ for an easy reason: any polynomial in $\mathcal{P}_3(\mathbb{R})$ has a form

  $$ax^3 + bx^2 + cx + d = a \cdot x^3 + b \cdot x^2 + c \cdot x + d \cdot 1$$

  which is a linear combination of the four vectors above. To show that they are linearly independent, we need to prove that $ax^3 + bx^2 + cx + d = 0$ identically (i.e. holds for all $x \in \mathbb{R}$) implies $a = b = c = d = 0$. There are several ways to prove this:

  - Use the following theorem; degree n *nonzero* polynomial has at most n zeros.
  - You can make many (in fact, four is enough) equations in a, b, c, d by plugging in different values of x, e.g. $x = 1$ gives $a + b + c + d = 0$, and solve the linear system to conclude $a = b = c = d = 0$. Surprisingly, you

don't need to choose x carefully - *any* four x would be enough. (If you want to know why, google *Vandermonde matrix*.)

- Another way is to use derivatives: if you have $p(x) = ax^3 + bx^2 + cx + d = 0$ identically, then we also have $p'(x) = 3ax^2 + 2bx + c = 0$. And keep doing this until we get $p'''(x) = 6a = 0$, which gives $a = 0$. Now by going backward, we get $b = c = d = 0$. This argument is neat, except for one point: this argument does not work over *finite fields* (we'll not treat this in Math110 though).

- Linear maps are *structure preserving maps*. Vector space is a set with two operations $+ : V \times V \to V$ and $\cdot : F \times V \to V$, they are "compatible" each other (satisfy the axioms). Then the linear maps are nothing but a map $L : V \to W$ that preserves vector spaces structures: additions and scalar multiplications. Especially, they always send $0_V \in V$ to $0_W \in W$ (Theorem 3.10).

  If you take Math113 or any abstract/abstracter/abstractest algebra courses, then the "structure preserving maps" are usually called *homomorphisms* for general algebraic structures. Bijective homomorphisms are called *isomorphisms*.

2 Problems

- Recommended problems: 2B.3, 2B.6, 2C.5, 2C.12, 2C.13, 3A.8, 3A.9.

- Additional problems:

  1. Let U, V, W be finite dimensional vector spaces. Prove or disprove the following identity:

  $$\dim((U + V) \cap W) = \dim(U \cap W) + \dim(V \cap W).$$

     What is the analogous equation for finite sets? Can you prove/disprove it?

  2. For 2.C.15, can we replace $>$ with $\geq$? Hint: the anwer is no. Find an example with $\dim V_1 + \dim V_2 + \dim V_3 = 2 \dim V$ and $V_1 \cap V_2 \cap V_3 = \{0\}$. (Hint: Try $V = \mathbb{R}^3$.)

  3. (a) Prove that

  $$(1, 2, 0, 0), (0, 1, 2, 0), (0, 0, 1, 2), (0, 0, 0, -1)$$

form a basis of $\mathbb{R}^4$.

(b) Prove that

$$x^3 + 2x^2, x^2 + 2x, x + 2, -1$$

form a basis of $\mathcal{P}_3(\mathbb{R})$.

(c) Prove that the map

$$ax^3 + bx^2 + cx + d \mapsto (a, b, c, d)$$

gives a bijective linear map from $\mathcal{P}_3(\mathbb{R})$ to $\mathbb{R}^4$.

(d) (*) Let $L : V \to W$ be a bijective linear map between two vector spaces V and W over $\mathbb{R}$. Show that $\{v_1, \ldots, v_n\}$ is a basis of V if and only if $\{L(v_1), \ldots, L(v_n)\}$ is a basis of W. Now redo the previous questions again.