Week 5, September 27

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1 Discussion notes

- Comments for the previous quiz: $\therefore$ is *therefore*, $\because$ is *because*. Do not confuse them (or just write "therefore" or "because")! Also, $A \subset B$ includes the possibility where $A = B$; this is mentioned in th e lecture and we'll adapt the convention (instead of using $\subseteq$).

- Isomorphisms: Bijective linear map between to vector spaces is called an isomorphism. One useful fact is that, a linear map $T : V \to V$ is bijective if and only if it is injective, if and only if it is surjective (thanks to FTLM). In other words, to prove a linear map T from V to itself is bijective, you only need to prove "half" of it (injectivity or surjectivity). Try Exercise 3D.20.

  As in the lecture, if $T : V \to W$ is a bijective linear map, we can prove that the inverse $T^{-1} : W \to V$ is also linear. Giving an isomorphism $T : \mathbf{F}^n \to V$ (for a vector space over $\mathbf{F}$ of dimension n) is same as choosing a basis of V: see where the standard basis vectors $\mathbf{e}_i$ maps to. In particular, any finite dimensional vector spaces over $\mathbf{F}$ of same dimension are isomorphic, and the only "invariant" of vector spaces preserved under isomorphisms is the dimension.

  Then you should ask this question: why we study vector spaces other than $\mathbf{F}^n$, if they are all isomorphic? There could be many answer for this, but for now, I'd just say that there are some "good" basis for special purposes. For example, you'll learn about *orthogonal* basis later, and these give nice bases for approximations. (Try to google Legendre polynomial.)

- Change of basis: For a linear map $T : V \to W$ and bases β, γ of V, W, we have a matrix representation $\mathcal{M}(T)_\beta^\gamma$. If we change β or γ to some other basis, what is a relation between original and new matrix representations?

  To answer this, we need to understand what is a *change of basis matrix*. It is just a matrix representation of an identity map $I : V \to V$, but with *different* basis for the source V and target V: $\mathcal{M}(I)_{\beta}^{\beta'}$. In other words, it describes how can we write a vector in V in terms of β', originally in terms of β. If we change back to β, then the corresponding matrix representation $\mathcal{M}(I)_{\beta'}^{\beta}$ is an inverse of $\mathcal{M}(I)_{\beta}^{\beta'}$.

Recall that composition of linear maps correspond to a multiplication of matrices: for $V \xrightarrow{T} W \xrightarrow{S} U$, we have (3.81)

$$\mathcal{M}(ST)_{\beta}^{\delta} = \mathcal{M}(S)_{\gamma}^{\delta}\mathcal{M}(T)_{\beta}^{\gamma}.$$

Here β, γ, δ are bases of V, W, U, respectively. Now, apply this to the case $(V, \beta') \xrightarrow{I} (V, \beta) \xrightarrow{T} (W, \gamma)$, we get

$$\mathcal{M}(T)_{\beta'}^{\gamma} = \mathcal{M}(T)_{\beta}^{\gamma}\mathcal{M}(I)_{\beta'}^{\beta} = \mathcal{M}(T)_{\beta}^{\gamma}(\mathcal{M}(I)_{\beta}^{\beta'})^{-1}$$

Similarly, by considering the case $(V, \beta) \xrightarrow{T} (W, \gamma) \xrightarrow{I} (W, \gamma')$, we have

$$\mathcal{M}(T)_{\beta}^{\gamma'} = \mathcal{M}(I)_{\gamma}^{\gamma'}\mathcal{M}(T)_{\beta}^{\gamma}.$$

In other words, changing basis of a target (resp. source) of a linear map corresponds to multiplying a change of basis (resp. inverse of it) on the left (resp. on the right). If we apply this twice, we can get the change-of-basis formula for the maps $T : V \to V$ (3.84):[1]

$$\mathcal{M}(T)_{\beta'}^{\beta'} = \mathcal{M}(T)_{\beta}^{\beta'}(\mathcal{M}(I)_{\beta}^{\beta'})^{-1} = \mathcal{M}(I)_{\beta}^{\beta'}\mathcal{M}(T)_{\beta}^{\beta}(\mathcal{M}(I)_{\beta}^{\beta'})^{-1}.$$

- Dual[2] space: $V' = \mathcal{L}(V, \mathbf{F})$. For a given basis $\beta = \{v_1, \ldots, v_n\}$ of V, we define a *dual basis* $\beta' = \{\varphi_1, \ldots, \varphi_n\}$ of β such that $\varphi_i(v_j) = 1$ if $i = j$, and 0 otherwise. This gives an isomorphism $V \to V'$ via $v_i \mapsto \varphi_i$, but this DOES depend on the choice of basis[3] (see the additional problem below). In fact, you should consider a dual vector space V' as a "mirror image" of V, not as something similar to V.

  For $T : V \to W$, we define a dual map $T' : W' \to V'$ as $T'(\varphi)(v) := \varphi(Tv)$ for $\varphi \in W'$ and $v \in V$, or more simply $T'\varphi := \varphi \circ T$ (3.118). In other words, this is defined as a "precomposing" map:

$$V \xrightarrow{T} W \xrightarrow{\varphi} \mathbf{F}.$$

[1] To match with (3.84), put $\beta = (u_1, \ldots, u_n)$, $\beta' = (v_1, \ldots, v_n)$ and $C = \mathcal{M}(I)_{\beta}^{\beta'}$, $A = \mathcal{M}(T)_{\beta}^{\beta}$, $B = \mathcal{M}(T)_{\beta'}^{\beta'}$. Then the above equation is $B = CAC^{-1} \Leftrightarrow A = C^{-1}BC$.

[2] The word *dual* is related to number 2.

[3] Most of the mathematicians (at least me) prefer "canonical" things, e.g. things that do not depend on the choice of basis.

The map $T \mapsto T'$ gives an isomorphism between two vector spaces $\mathcal{L}(V, W) \to \mathcal{L}(W', V')$ (Exercise 3F.18). Try to check 3.120 (algebraic properties of dual maps) on your own. Now fix bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ of V and W, respectively. Then we have an isomorphism $\mathcal{L}(V, W) \simeq \mathbf{F}^{m,n}$ given by the corresponding matrix representation $T \mapsto \mathcal{M}(T)^{\gamma}_{\beta}$ (3.71). Let $\beta' = \{\varphi_1, \ldots, \varphi_n\}$ and $\gamma' = \{\phi_1, \ldots, \phi_m\}$ be dual bases of β and γ (for V' and W'). Then we have another isomorphism $\mathcal{L}(W', V') \to \mathbf{F}^{n,m}$, given by $S \mapsto \mathcal{M}(S)^{\beta'}_{\gamma'}$. Now the most important fact about dual maps can be summarized into this one "commutative diagram":

$$\begin{array}{ccc}
\mathcal{L}(V, W) & \xrightarrow{T \mapsto \mathcal{M}(T)^{\gamma}_{\beta}} & \mathbf{F}^{n,m} \\
{\scriptstyle T \mapsto T'}\big\downarrow & & \big\downarrow{\scriptstyle M \mapsto M^{\intercal}} \\
\mathcal{L}(W', V') & \xrightarrow[S \mapsto \mathcal{M}(S)^{\beta'}_{\gamma'}]{} & \mathbf{F}^{m,n}
\end{array}$$

The diagram is "commutative" in the sense that following arrows $\rightarrow\downarrow$ gives the same result as $\downarrow\rightarrow$:

$$\mathcal{M}(T')^{\beta'}_{\gamma'} = (\mathcal{M}(T)^{\gamma}_{\beta})^{\intercal}.$$

In other words, matrix representation of a dual map is a transpose of the original matrix representation, which is the *true* meaning of matrix transpose. Try to check this manually when $\dim V = \dim W = 2$; again, do this once in your life (3.132 gives a proof of general case).

Since we can define a dual space for any given vector space, we can consider *dual of dual*, i.e. double dual $V'' = (V')'$. Then we have a *natural* map from $V \to V''$, defined as

$$v \mapsto (f \mapsto f(v))$$

for $v \in V$ and $f \in V'$ (notations might be confusing; try to understand how the map is defined and why this is a linear map). Nice thing about this map is that we don't need to choose a basis to define it; it is a *natural* (or canonical) map that we can define.

2 Problems

- Recommended problems: 3D.3, 3D.9, 3D.21, 3D.22, 3D.24 (very important fact!), 3F.3 (you need this to guarantee an existence of a dual basis) 3F.4, 3F.9, 3F.11, 3D.12

Problems

- Additional problems:

1. (From the quiz) Let $T : V \longrightarrow V$ be a linear map between finite dimensional vector spaces, such that $T^2 = 0$.

   (a) Prove that $\text{range}(T) \subseteq \text{null}(T)$ (Review!).

   (b) Find an example of V and T such that $T^2 = 0$ and $\text{range}(T) = \text{null}(T)$.

   (c) Find an example of V and T such that $T^2 = 0$ and $\text{range}(T) \neq \text{null}(T)$.

   (d) When $\dim V$ is even, prove that there exists T with $T^2 = 0$ and $\text{range}(T) = \text{null}(T)$.

   (e) When $\dim V$ is odd, prove that there is no T with $T^2 = 0$ and $\text{range}(T) = \text{null}(T)$.[4]

2. (Rank of a linear map) We know how to define a rank of a *matrix*. Now, let's define a rank of a *linear map* $T : V \longrightarrow W$ as a rank of a corresponding matrix representation $\mathcal{M}(T)_\beta^\gamma$, with respect to *some basis* β, γ. Explain why it does not depend on a choice of β or γ.

3. (Change of basis of dual basis)

   (a) Let V be a finite dimensional vector space and β_1, β_2 be bases of V, and $C = \mathcal{M}(I)_{\beta_1}^{\beta_2}$ be the change of basis matrix. Let β_1' and β_2' be the dual bases of β_1 and β_2, respectively, and $C' = \mathcal{M}(I)_{\beta_1'}^{\beta_2'}$ be the change of basis matrix. What is a relation between C and C'?

   (b) Let $T_1 : V \longrightarrow V'$ be the isomorphism determined by β_1 and β_1', and $T_2 : V \longrightarrow V'$ similarly by β_2 and β_2'. What are $\mathcal{M}(T_1)_{\beta_1}^{\beta_1'}$ and $\mathcal{M}(T_2)_{\beta_2}^{\beta_2'}$? Can you express $\mathcal{M}(T_1)_{\beta_2}^{\beta_2'}$ in terms of C in (a) above?

[4] Imagine you are proving the following statement without FTLM: show that there's no 3 by 3 matrix A such that $\text{null}(A) = \text{range}(A)$. One of the main point of learning *abstract* linear algebra is to realize that working "abstractly" actually simplifies situations.