Week 6, October 4

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1 Discussion notes

- More dual space: We discussed about dual spaces, dual basis, and double dual last week. The definitions can be found in the previous week's note. It can be summarized as:
  - Dual map is defined via composition: $V \xrightarrow{T} W \xrightarrow{\varphi} F$, $T'(\varphi) := \varphi \circ T$.
  - Matrix representation of T' and T are transpose of each other.

  We did explicit two examples: Problem 5 of mock test and Exercise 3F.15 in the book. All you need to know are the definitions.

  Now, our goal is to understand the relation between T and T'. We learned that their matrix representations are transpose of each other (again, see the last week's note for the precise statement), we can ask about other properties. For example, can we relate range(T) with "something" of T'? What can we say about T' if T is injective?

  Before we give a "mathematical" answer, here's a "philosophical" answer first. This is a table of "dual" concepts with relevant theorems that I'm going to explain:

Concept	(Concept)′	Theorems
zero space	whole space	3.127
null space	range	3.128, 3.130
injective	surjective	3.129, 3.131
span	intersection	Exercise 3F.22
(subspace)	(quotient)	(Exercise 3F.33)

Table 1: Dual concepts.

To do actual math, we define *annihilator* of a subspace $U \subset V$, as follows:

$$U^0 = \{\varphi \in V' : \varphi(u) = 0, \forall u \in U\},$$

which is a subspace of V'. In other words, U^0 is a subspace of V' (not V) that "annihilates" all the vectors in U (prove that it is actually a subspace of V yourself, which is 3.124). For the inclusion map $i : U \to V$, we have $U^0 = \text{null}(i')$ (this follows from "definition"; try to figure out why), and applying FTLM to $i' : V' \to U'$ gives a dimension formula

$$\dim U^0 = \dim V - \dim U.$$

Now we explain the Table 1. Dual concepts of the first two rows are related by annihilators.

Theorem (3.127)**.**

1. $U = \{0\}$ if and only if $U^0 = V'$.
2. $U = V$ if and only if $U^0 = \{0\}$.

In other words, one side is a zero space if and only if the other side is the whole space.

Theorem (3.128, 3.130)**.** Let $T : V \to W$ be a linear map.

1. $\text{null}(T') = (\text{range}(T))^0$.
2. $\text{range}(T') = (\text{null}(T))^0$.

Using the above theorems, we can prove a theorem for the fourth row:

Theorem (3.129, 3.131)**.** Let $T : V \to W$ be a linear map.

1. T is injective if and only if T' is surjective.
2. T is surjective if and only if T' is injective.

You can prove these without referring to annihilators: see the additional problems below. Also, I may not say anything about the last row, but if you want to know the details, read the chapter 3.E (which we skipped) and do the Exercise 3F.33.

At last, please do the Exercise 3F.32 on the double dual: it contains the details that we haven't covered. Upshot is that V is isomorphic to V'' *naturally*, where the isomorphism can be defined without choosing a basis. Also, under the isomorphism, $U \subset V$ and $U^{00} = (U^0)^0$ are identified.

- Polynomials: This chapter will be used as a toolkit for chapter 5, eigenvalues and eigenvectors. You don't need to know the proof of the fundamental theorem of algebra (4.12), but the statement itself is very important. I recommend you to do the Exercise 7, especially using a linear algebra. (There's a non-linear algebra proof of it, e.g. using Lagrange interpolation.)

2 Problems

- Recommended problems: 3F.15, 3F.20, 3F.22, 3F.32 (Must!)
- Additional problems:

1. (More duals) Let $V = \mathcal{P}_1(\mathbb{R})$.
   (a) Prove that $\beta = \{t, 2t - 1\}$ is a basis of V.
   (b) Prove that $\gamma = \{\phi_1, \phi_2\}$ is a basis of V', where these linear functionals are defined by $\phi_1(p(x)) = p'(1)$ and $\phi_2(p(x)) = p(2)$.
   (c) Express a dual basis β' of β in terms of γ.
2. (Integration without integration) Let $V = \mathcal{P}_2(\mathbb{R})$ and consider a linear functional $\varphi : V \to \mathbb{R}, \varphi(p) = \int_{-1}^{1} p(x)\mathrm{d}x$.
   (a) Let $\beta = \{1, x, x^2\}$ be a basis of V. Find a dual basis $\beta' = \{\varphi_1, \varphi_2, \varphi_3\}$ of β. (Answer is in the one of the exercises in 3F.)
   (b) Since β' is a basis of $\mathcal{P}_2(\mathbb{R})'$ and $\varphi \in \mathcal{P}_2(\mathbb{R})'$, we write φ as a linear combination of $\varphi_1, \varphi_2, \varphi_3$. Find the corresponding coefficients.
   From this, you can compute an integral of degree ≤ 2 polynomial with derivatives!
   (c) Let $\gamma = \{\phi_{-1}, \phi_0, \phi_1\}$ where $\phi_a \in \mathcal{P}_2(\mathbb{R})'$ is an evaluation functional: $\phi_a(p) = p(a)$. Prove that γ is a basis of $\mathcal{P}_2(\mathbb{R})'$.
   (d) Again, if γ is a basis, you can express φ as a linear combination of $\varphi_{-1}, \varphi_0, \varphi_1$. Find the corresponding coefficients.
   From this, you can compute an integral of degree ≤ 2 polynomial with its values!

   Such an idea is used in a numerical integration, where we can approximate an integral of an "arbitrary" function only using its values at certain points. For example, see this post on various approximation techniques.

3. Prove theorem 3.129 and 3.131 without using annihilators. Here's a hint for the direction (T is surjective implies T' is injective): Assume T is surjective. Let $\varphi \in W'$ and assume $T'(\varphi) = 0$. This means $\varphi(Tv) = 0$ for all $v \in V$, and we want to prove that $\varphi = 0$, i.e. $\varphi(w) = 0$ for all $w \in W$. However, this is true because T is...