Week 8, October 18

Seewoo Lee

1 Discussion notes

- Based on the midterm course evaluation and the exam itself, I decided to focus on the homework problems. More precisely, concepts will be explained *through* the problems. Please *read* the homework problems before come to the following discussions - better if you have some ideas for the problems.
- Theorems that you need to know:
  - 5.7: equivalence conditions for being an eigenvalue
  - 5.11: eigenvectors for distinct eigenvalues are linearly independent
  - 5.22: existence and uniqueness of a minimal polynomial, with degree $\leq \dim V$
  - 5.27: eigenvalues are zeros of the minimal polynomial
- (5B.1, 5B.4) On eigenvalues and eigenvectors. Second one is a generalization of the first one over $\mathbb{C}$ (over $\mathbb{R}$, $\Leftarrow$ still holds, but $\Rightarrow$ may not). Here's a fancy (constructible) solution of 5B.1 ($\Rightarrow$): assume 9 is an eigenvalue of T^2 and v be an eigenvector, so $T^2v = 9v$. If $Tv = 3v$, then we're done. Otherwise, define $w = Tv - 3v$, then $w \neq 0$ and check that $Tw = T^2v - 3Tv = 9v - 3Tv = (-3)(Tv - 3v) = (-3)w$, and we obtain an eigenvector for (-3).
- (5B.3, 5B.8) How to compute the minimal polynomial. For 5B.3, it is easier to find the eigenvalues ($\lambda = 0, n$) first and to check if the "minimal candidate" $p(x) = x(x - n)$ works or not (it works!). For 5B.8, it does not have eigenvalues in $\mathbb{R}$, hence you may need to compute it directly, by seeking the relations between I, T, T^2 (the degree is at most two, since it is a 2 by 2 matrix).
- (5B.11) Cayley–Hamilton theorem. This problem tell you how to compute a minimal polynomial of a linear operator defined on a vector space of dimension 2. There's a version for n by n matrices: see the additional problem below.

- Minimal polynomial of
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
is $(x-1)^2 = x^2 - 2x + 1$.
- (5B.6, Real vs Complex) Linear operators on a real vector space may not have an eigenvalue. However, linear operator on a complex vector space always have an eigenvalue (by the fundamental theorem of algebra, 4.12 and 4.13).

2 Problems

- Recommended problems: 5A.10, 5A.24, 5B.20 (not in HW, but still good one)
- Additional problems:

1. Find a minimal polynomial of
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
2. (Problem 5B.8, geometrically)
   (a) Let $T = T_\theta \in \mathcal{L}(\mathbb{R}^2)$ be the operator of clockwise rotation by $\theta \neq 0, \pi$, then the minimal polynomial of T is $x^2 - 2(\cos\theta)x + 1$. Prove this by expressing $T^2 v$ as a linear combination of v and Tv. (It might be easier to express Tv as a combination of v and $T^2 v$)
   (b) Explain why $T_{2\theta} = T_\theta^2$. Using this, prove the double angle formula $\cos 2\theta = 2\cos^2 - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
3. (Cayley–Hamilton for 3 by 3 matrices) In general, it is known that an n by n matrix A satisfy the equation $c_A(A) = 0$, where $c_A(x) := \det(xI - A)$ is the *characteristic polynomial* (which we'll learn later). The problem 5B.11 (a) is to prove this for $n = 2$. Now, do this for $n = 3$. (Caution: this takes a lot of time!)