Week 9, October 25

Seewoo Lee

1 Discussion notes

- Theorems that you need to know:
  - Theorem 5.41: Eigenvalues of an upper-triangular matrix are diagonal entries.
  - Theorem 5.44: A linear map can be written as an upper-triangular matrix if and only if minimal polynomial factors completely (as linear factors) over a base field $\mathbf{F}$.
  - Theorem 5.47 (Corollary of 5.44): When $\mathbf{F} = \mathbb{C}$, every linear map can be represented as an upper-triangular matrix.
  - Theorem 5.55: Equivalent conditions for diagonalizability, in terms of eigenspaces. The main content of the theorem is whether eigenspaces span whole V or not, but not about their intersections - intersection of distinct eigenspaces are always zero, independent of the diagonalizability of T.
  - Theorem 5.58: A linear map is diagonalizable if it has $(\dim V)$-many distinct eigenvalues
  - Theorem 5.62: A linear map is diagonalizable if and only if the minimal polynomial has no repeated roots.
  - Theorem 5.76: Simultaneous diagonalizability $\Leftrightarrow$ commutativity
  - Theorem 5.80: Simultaneous triangulizability $\Leftrightarrow$ commutativity

  I didn't have enough time to discuss 5E (the last two theorems). I believe you can read it yourself, but let me know if you have any questions about the section.

- The latest quiz question Q2 is a really nice example on triangulizability and diagonalizability. Although we consider it over a complex vector space, we know that the minimal polynomial is $p(z) = z^{n-1}(z-1)$ and it even factors linearly over $\mathbb{R}$. Hence it is triangulizable both over $\mathbb{R}$ and $\mathbb{C}$. The matrix representation with respect to the basis $(e_1, \ldots, e_n)$ is *not upper-triangular*:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

rather it is *lower-triangular*. But Theorem 5.47 tell us that we should be able to write it as an upper-triangular matrix with respect to some other basis, and try to find a corresponding basis. In fact, we can choose $\beta = (e_n, e_{n-1}, \ldots, e_1)$ so that we get an upper-triangular matrix. This is related to the exercises 5C.10 and 5C.11. Especially, we have the following identity in a really bad notation:

$$[T]_{\rotatebox{180}{β}} = \rotatebox{180}{$[T]_\beta$}$$

Here $\rotatebox{180}{$\beta$} = (v_n, \ldots, v_1)$ when $\beta = (v_1, \ldots, v_n)$, and the right hand side is a matrix obtained by "rotating" a matrix $[T]_\beta$ 180 degree. (NEVER use this notation in an exam!) Also, this is the reason why we only care about *upper*-triangular matrices: upper-triangulizability and lower-triangulizability is equivalent.

For diagonalizability, it is diagonalizable if and only if $n = 2$. This can be seen from Theorem 5.55 or 5.62. In view of Theorem 5.55, one can compute the eigenspaces for $\lambda = 0, 1$, which are

$$E(0, T) = \operatorname{span}(e_{n-1} - e_n), \quad E(1, T) = \operatorname{span}(e_n)$$

which are both 1-dimensional. Hence T is diagonalizable if and only if $n = \dim V = \dim E(0, T) + \dim E(1, T) = 2$. In view of Theorem 5.62, the minimal polynomial $p(z) = z^{n-1}(z-1)$ has no repeated roots if and only if $n - 1 = 1 \Leftrightarrow n = 2$, getting the same answer.

- (5C.2, 5C.3) Try small matrices first ($n = 3$), and write a proof for general n. This is a good exercise for writing a proof with Σ.
- (5C.8) Triangulizability highly depends on the base field $\mathbf{F}$.
- (5C.9) You should be able to answer this without using a pen!

Problems

- (5D.1, 5D.11, 5D.14a) Life tip: to find a counter example related to diagonalization, first consider matrices of the form

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

  For example, for 5D.11 consider a matrix containing the above matrix as a submatrix.

- (5D.3, 5D.4, 5D.5) Use Theorem 5.55. For 5D.3, range(T) has to be the same as "some" direct sum of eigenspaces.

- (5D.21) One usual application of diagonalization: we can easily compute the power of a matrix. This problem also appears in the cover of the book!

- (5E.3, 5E.5) Use definition.

2 Problems

- Recommended problems: 5C.11, 5D.14 (for (b), use 5D.15)

- Additional problems:

  1. (Preview of Jordan canonical form) The matrix

  $$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

     is not diagonalizable. But, you can still compute J^n - what is it? Try $J^2, J^3, \ldots$ and guess the answer.

     Using the above result, compute

  $$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^{2024}.$$

     Hint: Check

  $$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}.$$