1. (Principle of Inclusion and Exclusion) Let $A = \{1, 2, 3, 4, \ldots, 60\}$.
   (a) How many numbers in A that are multiples of 3?
   (b) How many numbers in A that are multiples of 5?
   (c) How many numbers in A that are multiples of 3 and 5?
   (d) How many numbers in A that are multiples of 3 or 5?

2. (Countings) Count the followings.
   (a) You have seven textbooks and five novels. In how many ways can three of these textbooks and two of these novels be selected and arranged on a bookshelf?
   (b) Same as (a), but what if novels are adjacent each other?
   (c) Same as (a), but what if they are arranged in order of textbook-novel-textbook-novel-textbook?
   (d) How many permutations of the letter MISSISSIPPI?
   (e) How many 6-letter words that contains Z exactly once?
   (f) How many 6-letter words that contains Z at least once?

3. (Count in multiple ways) Let's toss a coin 5 times.
   (a) What is the total number of possible outcomes?
   (b) For each $0 \leq k \leq 5$, how many possible outcomes are there with exactly k heads?
   (c) For each $0 \leq k \leq 5$, how many possible outcomes are there that starts with exactly k consecutive heads?
   (d) Using (b) and (c), can you answer (a) again? Can you generalize this?

4. (Pigeonhole principle)
   (a) Show that if there are 35 students in a class, then at least two have last names that begin with the same letter.
   (b) How many students are required to guarantee that at least two of them have both first and last names begin with the same letter (e.g. Travis Scott and Thomas Scanlon[1])?

[1] https://math.berkeley.edu/~scanlon/

1. (Principle of Inclusion and Exclusion) Let $A = \{1, 2, 3, 4, \dots, 60\}$.
   (a) How many numbers in A that are multiples of 3?
   (b) How many numbers in A that are multiples of 5?
   (c) How many numbers in A that are multiples of 3 and 5?
   (d) How many numbers in A that are multiples of 3 or 5?

(a) $60/3 = 20$.
(b) $60/5 = 12$.
(c) They are multiples of 15. So $60/15 = 4$.
(d) $20 + 12 - 4 = 28$.

2. (Countings) Count the followings.
   (a) You have seven textbooks and five novels. In how many ways can three of these textbooks and two of these novels be selected and arranged on a bookshelf?
   (b) Same as (a), but what if novels are adjacent each other?
   (c) Same as (a), but what if they are arranged in order of textbook-novel-textbook-novel-textbook?
   (d) How many permutations of the letter MISSISSIPPI?
   (e) How many 6-letter words that contains Z exactly once?
   (f) How many 6-letter words that contains Z at least once?

(a) ${}_7C_3 \times {}_5C_2 \times 5! = 42000$.
(b) ${}_7C_3 \times {}_5C_2 \times 4! \times 2! = 16800$.
(c) ${}_7C_3 \times {}_5C_2 \times 3! \times 2! = 4200$.
(d) $\frac{11!}{4! \times 4! \times 2!} = 34650$.
(e) 6×25^5.
(f) $26^6 - 25^6$.

3. (Count in multiple ways) Let's toss a coin 5 times.
   (a) What is the total number of possible outcomes?
   (b) For each $0 \leq k \leq 5$, how many possible outcomes are there with exactly k heads?
   (c) For each $0 \leq k \leq 5$, how many possible outcomes are there that starts with exactly k consecutive heads?
   (d) Using (b) and (c), can you answer (a) again? Can you generalize this?

   (a) $2^5 = 32$.
   (b) ${}_5C_k$.
   (c) There are only 1 possible outcome for $k = 5$ (HHHHH). For $0 \leq k \leq 4$, it starts with k Heads, then one Tail, and we only need to determine the last $5-(k+1) = 4-k$ tosses, which gives 2^{4-k}.
   (d) Summing the number of outcomes for each k gives the total number of outcomes. This gives

$$32 = {}_5C_0 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 1 + 2^0 + 2^1 + 2^2 + 2^3 + 2^4.$$

   In general, if we consider n coin tosses, the same argument gives

$$2^n = \sum_{k=0}^{n} {}_nC_k = 1 + \sum_{k=0}^{n-1} 2^k.$$

4. (Pigeonhole principle)
   (a) Show that if there are 35 students in a class, then at least two have last names that begin with the same letter.
   (b) How many students are required to guarantee that at least two of them have both first and last names begin with the same letter (e.g. Travis Scott and Thomas Scanlon[1])?

   (a) 35 pigeons and 26 pigeonholes (a to z), so at least two have last names that begin with the same letter. In fact, $27 = 26 + 1$ students are enough.
   (b) There are 26^2 possible outcomes of tuple of the starting letters of first and last names. So $26^2 + 1$ students are enough to have such two students.

[1] https://math.berkeley.edu/~scanlon/