1 Pigeonhole Principle

1. Consider a game of musical chairs that begins with n players and k chairs. At the end of each round, every player not in a chair is eliminated (one player per chair maximum) and a chair is removed.
If $n < k$, how many rounds until a player is eliminated?

If $n = k$?

If $n > k$?

Explain why musical chairs will always eventually have a winner regardless of the starting number of chairs and contestants.

2. In a box of crayons, there are 5 shades of blue, 5 shades of green, 5 shades of red, 5 shades of yellow, and 5 shades of brown.

How many crayons do you need to take out of the box before you can guarantee that you have taken out a blue crayon?

Before you have taken out both a blue crayon and a green crayon?

3. You wash seven pairs of socks. How many socks do you need to pull out of the dryer before you know you have a pair?

4. (Discrete Mathematics 6.2.18) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of the numbers add up to 16?

5. A hundred page scrapbook has 4 photo slots on odd pages and 5 photo slots on even pages. How many photos must be added to guarantee a slot will be double-filled?

6. (Discrete Mathematics 6.2.28) Show that in a group of 5 people (where every pair is either friends or enemies), there are not necessarily 3 mutual friends or 3 mutual enemies.

7. You have 45 hissing cockroaches in a tank - 28 of those are under three years old, 4 are missing an antenna, and 3 are both missing an antenna and under three years old.

How many cockroaches must you examine to guarantee you find one which is either under three years old or missing an antenna?

To guarantee you have found a cockroach that is both under three and missing an antenna?

To guarantee you have found a cockroach which is at least 3 years old and has both antennae?

8. You are in a room with five other people. Every person in the room armwrestles at least one other person. Show there are at least two people in the room that have armwrestled the same number of people.

9. Werewolf (The Game)! A werewolf is hunting members of a town. Every night, the werewolf slays one town member. There are two doctors (each with the power to revive exactly one person per game), two seers, and seven additional townsfolk. How many players must be slain for us to guarantee a doctor or seer has been slain?

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If $n < k$, how many rounds until a player is eliminated?

If $n = k$?

If $n > k$?

Explain why musical chairs will always eventually have a winner regardless of the starting number of chairs and contestants.

- $k - n + 1$
- 1
- 1
- Regardless of how many chairs you begin with, you can always reduce to a case where there are fewer chairs than people. By the pigeonhole principle, from that point forward, at least one person will be eliminated each round. Iterating, there will eventually be one seat remaining with more than one person. (Note: if you begin with one person, they are immediately the winner.)

2. In a box of crayons, there are 5 shades of blue, 5 shades of green, 5 shades of red, 5 shades of yellow, and 5 shades of brown.

How many crayons do you need to take out of the box before you can guarantee that you have taken out a blue crayon?

Before you have taken out both a blue crayon and a green crayon?

- 21
- 21

3. You wash seven pairs of socks. How many socks do you need to pull out of the dryer before you know you have a pair?

- 8

4. (Discrete Mathematics 6.2.18) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of the numbers add up to 16?

- 5

5. A hundred page scrapbook has 4 photo slots on odd pages and 5 photo slots on even pages. How many photos must be added to guarantee a slot will be double-filled?

- 451. (50 odd pages * 4 and 50 even pages * 5).

6. (Discrete Mathematics 6.2.28) Show that in a group of 5 people (where every pair is either friends or enemies), there are not necessarily 3 mutual friends or 3 mutual enemies.

- Let A, B, C, D, E be our people. The idea is to give each person exactly two friends and two enemies. Consider the following (where $\longleftrightarrow$ implies friendship and $\not\longleftrightarrow$ implies enemies).

$$\begin{array}{ccc} A & \longleftrightarrow & B \\ A & \longleftrightarrow & C \\ A & \not\longleftrightarrow & D \\ A & \not\longleftrightarrow & E \\ C & \longleftrightarrow & D \\ B & \longleftrightarrow & E \\ B & \not\longleftrightarrow & D \\ C & \not\longleftrightarrow & B \\ D & \longleftrightarrow & E \\ C & \not\longleftrightarrow & E \end{array}$$

  Our possible groups are:
  (A,B,C), (A,B,D), (A,B,E), (A,C,D), (A,C,E), (A,D,E), (B,C,D), (B,C,E), (B,D,E) and (C,D,E). It is clear that none of these groups are composed of three mutual friends or enemies.

7. You have 45 hissing cockroaches in a tank - 28 of those are under three years old, 4 are missing an antenna, and 3 are both missing an antenna and under three years old.

How many cockroaches must you examine to guarantee you find one which is either under three years old or missing an antenna?

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To guarantee you have found a cockroach which is at least 3 years old and has both antennae?

- 17. Inclusion-exclusion gives 29 who are less than 3 or missing an antenna. 45-29+1 =17.

- 43. 45-3+1=43.

- 30. 29+1=30.

8. You are in a room with five other people. Every person in the room armwrestles at least one other person. Show there are at least two people in the room that have armwrestled the same number of people.

- Suppose we have five boxes (labeled 1-5). Represent each person with a ball (5 people with you = 6 balls). Place each ball in the box corresponding to the number of people they have armwrestled. By the pigeonhole principle, 6 balls in 5 boxes means one box will have at least two balls.

9. Werewolf (The Game)! A werewolf is hunting members of a town. Every night, the werewolf slays one town member. There are two doctors (each with the power to revive exactly one person per game), two seers, and seven additional townsfolk. How many players must be slain for us to guarantee a doctor or seer has been slain?

- 10. The 7 townsfolk must be eliminated. Then in (the worst case scenario), we account for a doctor or seer being eliminated and revived twice, then we need another elimination. 7+2+1=10. (Note: in this scenario, the werewolf is NOT being treated as one of the townsfolk.)