Ramsey Numbers

1. You may recall that on Tuesday we found a configuration of five people, each of whom is friends or enemies with each other person, without any 'cliques' of three mutual friends or three mutual enemies. Over the course of this problem, we will discover the same is not possible for larger groups.

   (a) Pretend you are one of the six people. Prove using the pigeonhole principle that you must have at least three friends, or else at least three enemies.

   Solution: Besides yourself, there are five people. As well, there are two "holes": friends and enemies. Thus, one of the two categories must have at least $\lceil \frac{5}{2} \rceil = 3$ people in it.

   (b) Suppose now that you have three friends. (If enemies, the logic is the same.) Suppose as well that there are no cliques of three mutual friends. What can you conclude about your three friends?

   Solution: The intent of this question was perhaps communicated poorly, but essentially either there is a clique of three mutual friends, or else there is not. If there is not, then none of your three friends can be friends with each other, lest they form a clique with yourself. Thus, all three of your friends must be enemies, hence there is a clique of three mutual enemies. As the scenario where you have three enemies is functionally the same, we have shown that there must either be three mutual friends, or else three mutual enemies.

   (c) (Hint for the homework) Now suppose there are ten people, and you are one of them. Prove that you must have at least four friends, or else at least six enemies.

   Solution: Intuitively, either you have at least four friends, or you don't. If you don't, then yoou have fewer than four friends, meaning three or less. Thus, the remaining six or more people must all be your enemies.

   How this allows us to solve the problem in the homework is that we can now divide into two cases: the four friends case, and the six enemies case. In the first case, if you have four friends, either some pair are friends with each other (yielding a clique of three friends with yourself), or else nobody is friends with each other. But if nobody is friends with each other, then your four friends form a clique of four enemies. On the other hand, if you have six enemies, then we may apply our result from parts (a) and (b) to discover that there are either three mutual friends or three mutual enemies. If three mutual friends, we are done, and if three mutual enemies, then with yourself that makes four. So we have proven by examining every possible case that among ten people there is either a clique of three mutual friends, or else a clique of four mutual enemies.

Permutations and Combinations

1. You are planning a trip to Europe, and would like to see Amsterdam, Barcelona, Berlin, London, Paris, Rome, and Vienna. However, you only have the time and money to visit four of them.

   (a) How many possible collections of four cities could you visit?

   Solution: "Collections" implies that the internal ordering of the four cities doesn't matter. In other words, we want to use combinations. Thus, the answer is simply the number of ways to *choose* four items from a list of seven, or $C(7,4) = \frac{7!}{3!4!} = 35$.

   (b) How many possible routings are there for your trip? (Here, a routing refers to an order in which you might visit four cities).

   Solution: Here, since we are examining orderings, we instead want to use permutations. So the answer is $P(7,4) = \frac{7!}{3!} = 840$.

   (c) Suppose that you've settled on a specific set of four cities. How many routings are there just between those four?

   Solution: Once again, we want permutations, but we are permuting from a smaller set. In this case, there are only four possibilities to choose from, and we are ordering them all, so we get $P(4,4) = 4! = 24$.

   (d) Verify that your answer to (b) is your answer to (a) times your answer to (c)!

   Solution: A calculator will inform you that $35 \cdot 24 = 840$. Without one, you may still notice that $\frac{7!}{3!4!} \cdot 4! = \frac{7!}{3!}$, by virtue of the 4! canceling. Put verbally: the number of routings in total is the number of city choices, times the number of routings per choice.

2. Suppose that you are asked to make a password. In total we allow 102 characters: 26 uppercase letters, 26 lowercase, 10 digits, and 40 special characters. How many 10-character passwords are there with one uppercase letter, three digits, and two special characters?

Solution: Counting everything all at once can be tricky, so we're going to simplify by counting two separate quantities and multiplying them together.

First, consider some arbitrary password, like "(Mouse123)". We're going to "forget" some information here: namely, we're going to reduce each character to just what category it fits into. That is to say, (and) are special characters, M is an uppercase letter, "ouse" is lowercase, and 123 are numbers, so we might represent this as "&Ucccc###&". Here, & refers to a special character, U refers to an uppercase letter, c to a lowercase, and # to a number. Let's call this representation a "configuration." Many possible passwords will all share a configuration: another one with the same configuration might be, for instance, "$mileS100%". So there are some number of possible configurations, and for each configuration there is some number of possible passwords.

First portion: how many configurations are there? Well, we might begin by choosing one of the 10 characters to be the uppercase character. There are, intuitively, $C(10,1) = 10$ possible selections here, leaving 9 characters remaining. From there, you might choose three to be numbers, yielding a further $C(9,3)$ possibilities, and leaving 6 remaining character. Perhaps then you will select two to be special characters, for a further $C(6,2)$ possibilities. Then, the last four must be lowercase characters, i.e. $C(4,4) = 1$ possibility. So the total number of configurations is $C(10,1)C(9,3)C(6,2)C(4,4) = \frac{10!}{4!3!2!1!}$.

(An astute observer here might ask the question: what if we selected in another order? Would this change the value? As it turns out, it does not! Try this for yourself: choose characters in any order you like. Then, fully write out the expression, and cancel like terms. When the dust settles, you will invariable wind up with $\frac{10!}{4!3!2!1!}$.)

Second portion: In an arbitrary configuration, how many possible passwords are there? Here, things are simpler. Consider, for instance, the configuration from our example: "&Ucccc###&". The first character is a special character, so there are 40 possibilities there. The uppercase character has 26 possibilities, each lowercase also has 26, each number has 10, and the last spacial character has 40 as well. Thus, the number of passwords for this configuration is $40 \cdot 26 \cdot 26^4 \cdot 10^3 \cdot 40 = 40^2 \cdot 26^5 \cdot 10^3$. And you may observe (by commutativity of multiplication) that we can swap around elements of the configuration without changing this numbers - that is to say, all configurations have $40^2 \cdot 26^5 \cdot 10^3$ associated passwords.

Thus, putting it together: the total number of passwords is the number of configurations times the number of passwords per configuration, or the rather messy $\frac{10!}{4!3!2!1!} \cdot 40^2 \cdot 26^5 \cdot 10^3$.

3. Suppose there are ten friend groups, each with three people. Suppose also that there are fourteen (distinct) empty apartments, each with four (distinct) bedrooms. How many ways are there to assign people to bedrooms, assuming that each friend group gets one apartment, and each person gets one bedroom?

   Solution: Since the apartments are distinct, it does matter which friend group gets which. Thus, we want to use permutations to figure out how many assignments of apartments there are, giving $P(14,10)$ possible such assignments. Then, within each assignment, there are $P(4,3)$ possible room assignments, as each room is also distinct. Thus, the total number of possible assignments with the given stipulations is $P(14,10) \cdot P(4,3)$.

4. In this last problem, we are going to prove the (initially surprising) fact that $C(n,0) + C(n,1) + ... + C(n,n) = 2^n$.

   (a) Suppose I flip a coin n times. How many possible outcomes are there?

   Solution: Each time you flip the coin, you introduce two branching outcomes. Thus, the total number of outcomes after n flips is 2 times itself n times, or 2^n.

   (b) How many ways are there to get zero heads? What about exactly one head? Two heads?

   Solution: In general, the number of ways to to get m heads may be rephrased as the number of ways to *choose* m of the n coin flips to come up heads. In other words, $C(n,m)$. So there are $C(n,0)$ outcomes with zero heads, $C(n,1)$ with one head, $C(n,2)$ with two, and so on.

   (c) If I add up the ways to get each possible number of heads, I should account for all the possible outcomes. Using this fact in conjunction with parts (a) and (b), deduce the equation above.

Solution: Intuitively, we are counting the same quantity - the number of possible outcomes from flipping n coins - two different ways. The first way is to just compute it directly, as we did in part (a), giving 2^n. The second way is to break it down into cases: the case where we get zero heads, the case where we get one head, the case with two heads, and so on. If we add together the number of possibilities with each case, that should give the total number of outcomes. Thus, $C(n,0)+C(n,1)+\cdots+C(n,n)$ also describes the number of outcomes from flipping a coin n times. Since both this and 2^n count the same thing, they must both be equal to each other.