1. (a) How many 4 digit numbers that starts with 3?
   (b) How many ways are there to put 10 indistinguishable balls into 5 distinguishable boxes?
   (c) You randomly shuffle a standard deck of 52 poker cards, and draw 4 cards from it. What is the probability that you get all 4 aces?
   (d) How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, and $x_5 \geq 5$?
   (e) What is the probability that when you roll a fair die 8 times, you never get a multiple of 3?
   (f) How many numbers must be selected from the set $\{1,2,3,4,5,6,7,8,9,10\}$ to guarantee that at least one pair of these numbers add up to 7?

---

(a) $10^3 = 1000$.

(b) $C(5+10-1,10) = C(14,10)$.

(c) $1/C(52,4)$.

(d) Consider a new equation $(x_1 - 1) + (x_2 - 2) + (x_3 - 3) + (x_4 - 4) + (x_5 - 5) = 5$. $C(5+5-1,5) = C(9,5)$.

(e) For each roll, the probability of getting a number that is not a multiple of 3 is $4/6 = 2/3$. So the answer is $(2/3)^8$.

(f) Divide the numbers into the following groups:

$$\{1,6\},\{2,5\},\{3,4\},\{7\},\{8\},\{9\},\{10\}.$$

Now we want to guarantee that we choose two numbers from one of the first three groups. The worst possible scenario is choosing one from each group (first three), and choose 7, 8, 9, 10 - total 7 numbers with no pair add up to 7. Now one more selection guarantees a pair add up to 7, hence we need $7 + 1 = 8$.

2. (a) How many strings of length 10 with five upper case letters and five lower case letters, where no two upper case letters are adjacent and no two lower case letters are adjacent?
   (b) How many nonnegative integer solutions for the inequality $x_1 + x_2 + x_3 + x_4 \leq 12$?
   (c) Roll a die three times. What is the probability to get the total outcome of 7?
   (d) Show that whenever 5 cats and 5 dogs are seated around a circular table there is always an animal both of whose neighbors are cats.

---

(a) There are essentially two possible types: ULULULULUL or LULULULULU (U: upper, L: lower). Each has $26^5 \times 26^5 = 26^{10}$ possibilities, hence total $2 \times 26^{10}$.

(b) If we write $x_1 + x_2 + x_3 + x_4 = 12 - x_5$ with $x_5 \geq 0$, then it is equivalent to count the number of nonnegative integer solutions for the new equation $x_1+x_2+x_3+x_4+x_5 = 12$. There are $C(5+12-1, 12) = C(16, 12)$ solutions of them.

(c) If we write $x_1, x_2, x_3$ for each outcome, then we need to count the number of integer solutions for $x_1 + x_2 + x_3 = 7$ with $1 \leq x_i \leq 6$ for each $i$. By subtracting 1 from each variable, we get $(x_1-1)+(x_2-1)+(x_3-1) = y_1+y_2+y_3 = 4$ with $y_i = x_i-1$ and $0 \leq y_i \leq 5$. We can ignore the upper bounds, since nonnegativity gives $y_1 = 4-y_2-y_3 \leq 4$, and same for $y_2$ and $y_3$. Hence there are $C(3+4-1, 4) = C(6, 4)$ many ways to get the total outcome of 7, and the probaility is $C(6, 4)/6^3$.

(d) Number the seats around the table from 1 to 10, and think of seat 10 as being adjacent to seat 1. There are 5 seats with odd numbers and 5 seats with even numbers. If no more than 3 boys occupied the odd-numbered seats, then at least 3 cats would occupy the even-numbered seats, and vice versa. Without loss of generality, assume that at least 3 cats occupy the 5 odd-numbered seats. Then at least two of those cats must be in consecutive odd-numbered seats, and the animal sitting between them will have cats as both of its neighbors.

3. Suppose that I flip a coin six times.
   (a) What is the likelihood that I receive exactly two heads?
   (b) Is it more likely to receive exactly two heads if the first flip is heads, or if it is tails?
   (c) Explain intuitively why the one you chose is more likely.

---

(a) Among 6 flips, there are $C(6,2)$ many possible ways to get exactly two heads, and the probability is $C(6,2)/2^6$.

(b)

$$P(\text{two heads}|\text{first head}) = \frac{P(\text{two heads} \cap \text{first head})}{P(\text{first head})} = \frac{\mathrm{C}(5,1)/2^6}{2^5/2^6} = \frac{5}{2^6}$$

and

$$P(\text{two heads}|\text{first tail}) = \frac{P(\text{two heads} \cap \text{first tail})}{P(\text{first tail})} = \frac{\mathrm{C}(5,2)/2^6}{2^5/2^6} = \frac{10}{2^6}.$$

So it is more likely when the first flip is a tail.

(c) This is somehow open-ended question, but one explanation is just from the fact that the number of ways to get two heads from 5 flips ($= \mathrm{C}(5,2)$) is larger than getting one head ($= C(5,1)$).

4. (a) Let there be two independent events $E$ and $F$, with probabilities $P(E) = 0.5$ and $P(F) = 0.4$. What is $P(E \cup F)$?
   (b) If $E$ and $F$ are not necessarily independent, what is the possible range of $P(E \cup F)$?

---

(a) By independence, $P(E \cap F) = P(E)P(F) = 0.2$. Hence by the principle of inclusion-exclusion, $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.5 + 0.4 - 0.2 = 0.7$.

(b) We still have the principle of inclusion-exclusion, $P(E \cup F) = 0.9 - P(E \cap F)$. The maximum is 0.9 when $P(E \cap F) = 0$ (for example, $E$ and $F$ are disjoint events). Since $E \subseteq E \cup F$, we have $0.5 = P(E) \leq P(E \cup F)$ (we also have $0.4 = P(F) \leq P(E \cup F)$ from $F \subseteq E \cup F$, but this is a weaker inequality). Hence the minimum is 0.5, which may happens if $F \subset E$ and so $E \cup F = E$.