1. Suppose you flip a fair coin three time.
   (a) What is the probability that you get three heads?
   (b) What is the probability that you get exactly two heads?
   (c) What is the conditional probability that you get exactly two heads, given that at least one of the flips is a head?
   (d) What is the conditional probability that you get exactly two heads, given that at least one of the flips is a tail?
   (e) What is the conditional probability that you get at least one tail, given that exactly two flips are heads?
   (f) Now you toss the coin 99 times, and get 99 heads. You need to bet whether the next 100th toss is head or tail. What is your choice?

(a) $\frac{1}{8}$.

(b) $\frac{C(3,2)}{8} = \frac{3}{8}$.

(c) $P(H = 2|H \geq 1) = \frac{P((H=2)\cap(H\geq 1))}{P(H\geq 1)} = \frac{P(H=2)}{P(H\geq 1)} = \frac{3/8}{1-P(H=0)} = \frac{3/8}{1-1/8} = \frac{3}{7}$.

(d) $P(H = 2|T \geq 1) = \frac{P((H=2)\cap(T\geq 1))}{P(T\geq 1)} = \frac{P(H=2)}{P(T\geq 1)} = \frac{3/8}{1-P(T=0)} = \frac{3/8}{1-1/8} = \frac{3}{7}$.

(e) You can find that the probability is 1, without doing any calculation - if there are exactly two heads, then there must be a tail.

(f) Any choice you like. All tosses are independent.

2. True or False:

   (a) If E and F are mutually exclusive, then they are independent.

   (b) If E and F are independent, then they are mutually exclusive.

   (c) If E and F are mutually exclusive, then E^c and F^c are also mutually exclusive.

   (d) If E and F are independent, then E^c and F^c are also independent.

   (e) If E and F are independent and $P(F) > 0$, then $P(E|F) = P(E)$.

   (f) If E and F are independent and $P(E), P(F) > 0$, then $P(E|F) = P(F|E)$.

   (g) If E and F are mutually exclusive and $P(E), P(F) > 0$, then $P(E|F) = P(F|E)$.

   (a) False.

   (b) False. Actually, being mutually exclusive and independent are not related.

   (c) False. $E \cap F = \varnothing$ does not imply $E^c \cap F^c = \varnothing$. For example, consider $\Omega = \{1, 2\}$, $E = \varnothing$ and $F = \{1\}$.

   (d) True. This can be checked using the principle of inclusion and exclusion. We only need to show that $P(E^c \cap F^c) = P(E^c)P(F^c)$:

$$\begin{aligned} P(E^c \cap F^c) &= P((E \cup F)^c) \\ &= 1 - P(E \cup F) \\ &= 1 - (P(E) + P(F) - P(E \cap F)) \\ &= 1 - P(E) - P(F) + P(E \cap F) \\ &= (1 - P(E))(1 - P(F)) = P(E^c)P(F^c). \end{aligned}$$

   (e) True. Intuitively true (if two events are independent, then conditioning does not affect probability), and also mathematically true:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E).$$

   (f) False. $P(E|F) = P(F|E)$ if and only if $P(E) = P(F)$, and this is not true in general.

   (g) True. Both $P(E|F)$ and $P(F|E)$ are zero, since $E \cap F = \varnothing \Rightarrow P(E \cap F) = 0$.