1. (Conditional probabilities) Consider a standard deck of 52 cards. Shuffle the deck and draw two cards from it.
   (a) What is the probability of having two Queens?
   (b) What is the probability of having two Queens, given that the first card is a queen?
   (c) What is the probability of having two Queens, given that the second card is a queen?
   (d) What is the probability of having two Queens, given that the two cards have the same number/alphabet?
   (e) What is the probability of having two Queens, given that the first card has an alphabet on it (one of A, J, Q, K)?

(a) $\frac{4}{52} \times \frac{3}{51}$.

(b) $\frac{3}{51}$.

(c)

$$\begin{aligned} P(\text{both Q}|\text{second Q}) &= \frac{P(\text{both Q} \cap \text{second Q})}{P(\text{second Q})} \\ &= \frac{P(\text{both Q})}{P(\text{second Q})} \\ &= \frac{P(\text{both Q})}{P(\text{first Q} \cap \text{second Q}) + P(\text{first not Q} \cap \text{second Q})} \\ &= \frac{\frac{4}{52} \times \frac{3}{51}}{\frac{4}{52} \times \frac{3}{51} + \frac{48}{52} \times \frac{4}{51}} = \frac{3}{51}. \end{aligned}$$

(d) $\frac{\binom{4}{2}}{\binom{13}{1}\binom{4}{2}} = \frac{1}{13}$.

(e)

$$\begin{aligned} P(\text{both Q}|\text{first alphabet}) &= \frac{P(\text{both Q} \cap \text{first alphabet})}{P(\text{first alphabet})} \\ &= \frac{P(\text{both Q})}{P(\text{first alphabet})} \\ &= \frac{\frac{4}{52} \times \frac{3}{51}}{\frac{16}{52}} = \frac{3}{204}. \end{aligned}$$

2. (Bayes rule) Suppose that 4% of the patients tested in a clinic are infected with avian influenza. Furthermore, suppose that when a blood test for avian influenza is given, 97% of the patients infected with avian influenza test positive and that 2% of the patients not infected with avian influenza test positive. What is the probability that

   (a) a patient testing positive for avian influenza with this test is infected with it?

   (b) a patient testing positive for avian influenza with this test is not infected with it?

   (c) a patient testing negative for avian influenza with this test is infected with it?

   (d) a patient testing negative for avian influenza with this test is not infected with it?

(a) $\frac{0.97\times0.04}{0.97\times0.04+0.02\times0.96}$.

(b) $\frac{0.02\times0.96}{0.97\times0.04+0.02\times0.96}$.

(c) $\frac{0.03\times0.04}{0.98\times0.96+0.03\times0.04}$.

(d) $\frac{0.98\times0.96}{0.98\times0.96+0.03\times0.04}$.

3. (Bayesian learning) You have a coin, possibly unfair. It lands with head with probability $0 < p < 1$, and our goal is to estimate the probability from 100 flips.

   (a) Assume that we get 65 heads among 100 flips. What is a likelihood, as a function in p?

   (b) If you need to guess p, what would be your choice?

   (c) Let $f(p)$ be the function in (a). Find the derivative $f'(p)$ (Hint: you may remember keywords from 10A like product rule, logarithmic derivative, ...).

   (d) Find p with $f'(p) = 0$, which maximizes $f(p)$.

   (e) Compare the answer for (d) with with your guess in (b).

(a) $\binom{100}{65}p^{65}(1-p)^{35}$.

(b) "Natural" guess would be 65/100.

(c) $f'(p) = \binom{100}{65}(65p^{64}(1-p)^{35} - 35p^{65}(1-p)^{34}) = \binom{100}{65}p^{64}(1-p)^{34}(65(1-p) - 35p)$

(d) $65(1-p) - 35p = 65 - 100p = 0 \Rightarrow p = 65/100$. You can check that this maximizes $f(p)$, by using the second derivative test.

(e) We got the same answer.

This exercise is the most simplest case of so-called *Bayesian learning* with *maximum likelihood (ML) hypothesis*. In general, for a given data distribution D and a set of candidate hypotheses H, you seek for the "best" hypothesis $h = h_{\mathrm{ML}} \in H$ maximizing $P(D|H = h)$. In our case, D is the outcome of coin tosses, and H is the set of p's.