1. You roll two fair dices with *eight* sides on each, numbered from 1 to 8. Let X be the random variable indicates the total outcome.
   (a) What is the possible range of X?
   (b) For each value k in the above range, find the probability $P(X = k)$.
   (c) Compute the expected value $E[X]$ of X.
   (d) Can you answer (c) without doing (b)?
   (e) What is the variance $\mathrm{Var}[X]$?

(a) 2 to 16.

(b) There are total $8^2 = 64$ possible outcomes. You can simply enumerate the number of ways to get each number $k \in \{2, 3, \dots, 16\}$:

$$
\begin{aligned}
2 &\to (1,1), 1 \\
3 &\to (1,2), (2,1), 2 \\
4 &\to (1,3), (2,2), (3,1), 3 \\
&\vdots \\
8 &\to (1,7), (2,6), \dots, (6,2), (7,1), 7 \\
9 &\to (1,8), (2,7), \dots, (8,1), 8 \\
10 &\to (2,8), (3,7), \dots, (8,2), 7 \\
&\vdots \\
15 &\to (7,8), (8,7), 2 \\
16 &\to (8,8), 1
\end{aligned}
$$

and probabilities for each k above are the numbers divided by 64. It can be written as

$$
P(X = k) = \begin{cases} \frac{k-1}{64} & 2 \le k \le 9 \\ \frac{17-k}{64} & 10 \le k \le 16 \end{cases}.
$$

(c) the expected value is

$$\begin{aligned}E[X] &= \sum_{k=2}^{16} kP(X=k) = \sum_{k=2}^{9} k\cdot\frac{k-1}{64} + \sum_{k=10}^{16} k\cdot\frac{17-k}{64}\\ &= \frac{1}{64}\sum_{k=2}^{9} k^2 - \frac{1}{64}\sum_{k=2}^{9} k + \frac{17}{64}\sum_{k=10}^{16} k - \frac{1}{64}\sum_{k=10}^{16} k^2\\ &= \frac{1}{64}\left(\frac{9\cdot 10\cdot 18}{6}-1\right) - \frac{1}{64}\left(\frac{9\cdot 10}{2}-1\right) + \frac{17}{64}\left(\frac{16\cdot 17}{2}-\frac{9\cdot 10}{2}\right) - \frac{1}{64}\left(\frac{16\cdot 17\cdot 33}{6}-\frac{9\cdot 10\cdot 19}{6}\right)\\ &= 9.\end{aligned}$$

(d) Better way that (b) is using linearity of expected value. Let X_1 and X_2 be the random variables represent outcome of each die. Then $X = X_1 + X_2$ and we can easily check that $E[X_1] = E[X_2] = \frac{1}{8}(1+2+3+4+5+6+7+8) = \frac{9}{2}$. Thus $E[X] = E[X_1] + E[X_2] = 9$.

(e) Let X_1, X_2 as in (d). Variance of each random variable is:

$$\begin{aligned}E[X_1^2] &= \frac{1}{8}(1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2) = \frac{8\cdot 9\cdot 17}{8\cdot 6} = \frac{51}{2}\\ \Rightarrow \mathrm{Var}[X_1] &= E[X_1^2] - E[X_1]^2 = \frac{51}{2} - \frac{81}{4} = \frac{21}{4} = \mathrm{Var}[X_2].\end{aligned}$$

Since X_1 and X_2 are independent, we have $\mathrm{Var}[X] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] = \frac{21}{2}$.

In general, sum of two random variable could be much more complicated than the summands. However, computing expectation value can be easily done as (d), and also variance as (e) when two summands are independent.

2. True or False: for two random variables X and Y,

   (a) $E[X+Y] = E[X] + E[Y]$.

   (b) $E[XY] = E[X]E[Y]$.

   (c) If X and Y are independent, $E[XY] = E[X]E[Y]$.

   (d) If X and Y are independent, $\text{Var}[XY] = \text{Var}[X]\text{Var}[Y]$.

   (e) If X and Y are independent, $E[\max\{X,Y\}] = \max\{E[X], E[Y]\}$. (Hint: toss two fair coins. Let X be the number of heads, and Y be the number of tails.)

   (f) If $E[XY] = E[X]E[Y]$, then X and Y are independent. (Hint: consider the following situation: you randomly pick one of the four points $(-1,0)$, $(0,1)$, $(1,0)$ $(0,-1)$ with equal probability $1/4$, and let X, Y be x and y-coordinates of the picked point.)

(a) True.

(b) False in general. For example, consider any random variable X and set $Y = X$. Then the statement is False whenever $\text{Var}[X] \neq 0$.

(c) False. Instead, we have $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$. For a counterexample, set both X and Y be the number of heads when you toss a fair coin. Then $X = X^2$, and $\text{Var}[X^2] = \text{Var}[X] = \frac{1}{4}$ and $\text{Var}[X^2] \neq \text{Var}[X]^2$.

(d) False. Consider the example given in the hint. We have $E[X] = E[Y] = 1$. The maximum of two random variable is

$$\max\{X,Y\} = \begin{cases} 1 & X = Y = 1, \text{ probability} = \frac{1}{2} \\ 2 & (X,Y) = (2,0) \text{ or } (0,2), \text{ probability} = \frac{1}{2} \end{cases} \tag{1}$$

and the expected value is $E[\max\{X,Y\}] = \frac{3}{2} \neq \max\{E[X], E[Y]\} = 1$.

(e) False. Consider the example given in the hint. One can check that $E[XY] = E[X] = E[Y] = 0$. However, they are not independent, since

$$P(X = 1 \cap Y = 1) = 0 \neq \left(\frac{1}{4}\right)^2 = P(X = 1)P(Y = 1).$$