Expected Value, Variance, and Special Distributions

You may leave your answers in summation form where applicable.

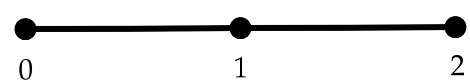
1. What is the expected number of heads in 10 flips for a coin biased 75/25 towards heads?

2. If we keep rolling a fair 6-sided dice until a 4 appears or we hit 8 rolls, what is the expected number of rolls?

3. What is the variance of the number of times a 4 appears in 8 rolls of a fair 6-sided dice?

4. What is the expected number of rolls required for a critical failure (1) or critical hit (20) on a 20-sided dice?

5. A random walk begins at 0 on the following interval.



Suppose that with probability 1 a particle at "0" moves to "1", and any particle at "1" moves to "0" with probability 1/2 and "2" with probability 1/2. What is the expected number of steps it will take for a particle started at 0 to get to 2?

6. We place 7 red balls and 3 blue balls in a box. We draw a ball from the box and replace it five times. We count how many times the ball we choose is blue ball.

- What special distribution does this random variable have?
- What is its expected value?
- What is its variance?

7. I am testing whiteboard markers in the third-floor classrooms of Evans. After testing them, I haphazardly throw them back in the pile. One out of every 12 markers works.

- What is the probability that the fourth marker I test is the first functional marker I find?
- What is the probability that I test at least 4 markers to find my first functional marker?
- What special distribution does this random variable have?

1.
$$E(X) = \sum_{k=0}^{10} k\binom{10}{k}(.75)^k(.25)^{10-k}$$

2.
$$E(X) = \sum_{k=1}^{8} kP(X = k)$$

where
$$P(X = k) = \begin{cases} \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{k-1} & \text{if } k < 8 \\ \left(\frac{5}{6}\right)^7 & \text{if } k = 8 \end{cases}$$

3.
$$\text{Var}(X) = E(X^2) - E(X)^2 = \left(\sum_{k=0}^{8} k^2\binom{8}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{8-k}\right) - \left(\sum_{k=0}^{8} k\binom{8}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{8-k}\right)^2$$

4.
$$E(X) = \sum_{k=1}^{\infty} k\left(\frac{18}{20}\right)^{k-1}\left(\frac{2}{20}\right)$$

5. We have that
$$E(X) = \sum_{k=0}^{\infty}(2k+2)\left(\frac{1}{2}\right)^{k+1}.$$

Each term in our sum represents the number of steps taken time the probability of taking that number of steps. The only difference between each path is how many times the particle goes back and forth between 0 and 1 before moving on to 2. For a path that goes from 0 to 1, back and forth between 1 and 0 k times, and then from 1 to 2 ($2k$+1 steps), the probability is $(1/2)^k$ for the looping between 1 and 0 times and additional 1/2 for the transition from 1 to 2.

6. Binomial (repeated Bernoullis).

$$E(X) = \sum_{k=0}^{5} k\binom{5}{k}\left(\frac{3}{10}\right)^k\left(\frac{7}{10}\right)^{5-k}.$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \left(\sum_{k=0}^{5} k^2\binom{5}{k}\left(\frac{3}{10}\right)^k\left(\frac{7}{10}\right)^{5-k}\right) - \left(\sum_{k=0}^{5} k\binom{5}{k}\left(\frac{3}{10}\right)^k\left(\frac{7}{10}\right)^{5-k}\right)^2.$$

7. The probability that the fourth test is the first success is

$$\left(\frac{11}{12}\right)^3\left(\frac{1}{12}\right).$$

The probability it takes at least 4 tests is

$$1 - P(X=1) - P(X=2) - P(X=3) = 1 - \frac{1}{12} - \left(\frac{11}{12}\right)\left(\frac{1}{12}\right) - \left(\frac{11}{12}\right)^2\left(\frac{1}{12}\right)$$

For part(a) we treat our random variable as geometric. For part (b) we view it as binomial.