1 Variance and Expected Value

Some helpful formulae:

- $E[aX + bY + c] = aE[X] + bE[Y] + c$
- $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$.
- $\text{Var}(aX + bY + c) = a^2\text{Var}(X) + b^2\text{Var}(Y)$, if X and Y are independent.
- A discrete uniform distribution with possible values $\{1, 2, \dots, n\}$ has variance $\frac{n^2-1}{12}$.

1. I flip a fair coin until it comes up heads, and give you an amount of money equivalent to the number of flips squared. To play the game, you must pay me \$3. How much will you win or lose on average playing this game?

   Solution: Let X be the amount of money you make, and let Y be the number of flips I make. Then $X = Y^2 - 3$, so $E[X] = E[Y^2] - 3$. Note that as $\text{Var}(Y) = E[Y^2] - E[Y]^2$, we can see $E[Y^2] = \text{Var}(Y) + E[Y]^2$. As Y is geometrically distributed with $p = 0.5$, this yields $E[Y^2] = 2 + 4 = 6$, hence $E[X] = 6 - 3 = 3$.

2. Alice enjoys playing a casual game of basketball with her friends. The number of 2-pointers she scores in a game is geometrically distributed with $p = \frac{1}{4}$, and the number of 3-pointers is binomially distributed with $p = \frac{1}{2}$ and $n = 4$. On average, how many points does Alice score in a typical game?

   Solution: Let X be the number of 2-pointers, and Y the number of 3-pointers. Then the score is $S = 2X + 3Y$, so $E[S] = 2E[X] + 3E[Y]$. $E[X] = \frac{1}{1/4} = 4$, and $E[Y] = 4\frac{1}{2} = 2$, so $S = 8 + 6 = 14$.

3. I roll an 11-sided die and a 4-sided die. I then double the second result, and subtract it from the first result. Call this difference S, what is $\text{Var}(S)$?

   Solution: Let X be the result of the first roll, and Y the result of the second. Then $S = X - 2Y = X + (-2)Y$, hence $\text{Var}(S) = \text{Var}(X) + 4\text{Var}(Y)$. As $\text{Var}(X) = \frac{121-1}{12} = 10$ and $\text{Var}(Y) = \frac{16-1}{12} = \frac{5}{4}$, we see $\text{Var}(S) = 10 + 4\frac{5}{4} = 10 + 5 = 15$.

2 Continuous Random Variables

1. Suppose I have a continuous random variable whose pdf is defined as follows:

$$p(x) = \begin{cases} x^2 & x \in [0,1) \\ \frac{1}{Cx} & x \in [1,e^2] \\ 0 & \text{Otherwise} \end{cases}$$

   What is C^2?

   Solution: Integrating x^2 from 0 to 1 gives $\frac{1}{3}$, so we need to have $\int_1^{e^2} \frac{1}{Cx} = \frac{2}{3}$. Thus $\frac{1}{C} \ln(x)|_1^{e^2} = \frac{1}{C}(2-0) = \frac{2}{3}$. So $C = 3$, and $C^2 = 9$.

2. Suppose I have a continuous random variable whose pdf is $p(x) = \frac{3}{100}x^2$ for $x \in [0,10]$. What is $P(1 \leq x \leq 3)$? Write your answer in decimal form.

   Solution: $P(1 \leq x \leq 3) = \int_1^3 \frac{3}{100}x^2 = \frac{1}{100}x^3|_1^3 = \frac{27}{100} - \frac{1}{100} = \frac{26}{100} = .26$.

3 Happy Pi Day!

Write out all the answers you got in order. :)
Solution: 3 14 15 9 .26