1. Let X be a uniform random variable on the interval $[-1, 3]$, i.e. the graph of PDF is a horizontal line.
   (a) Find pdf and cdf of X.
   (b) Find mean of X. (You can guess without doing any computation.)
   (c) Find median of X. (Again, you can guess without doing any computation.)
   (d) Find variance of X.
   (e) Try (a) - (d) with general interval $[a, b]$.

(a) • pdf: $f(x) = 1/4$ for $-1 \leq x \leq 3, f(x) = 0$ otherwise.
    • cdf: $F(x) = (x+1)/4$ for $-1 \leq x \leq 3$.

(b) 1.

(c) 1.

(d)

$$\int_{-1}^{3} (x-1)^2 \cdot \frac{1}{4} dx = \left[\frac{1}{12}(x-1)^3\right]_{-1}^{3} = \frac{4}{3}$$

(e) Here we assume $a < b$.

pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b, f(x) = 0$ otherwise.

- cdf: $F(x) = (x-a)/(b-a)$ for $a \leq x \leq b$.
- mean, median = $(a+b)/2$.
- variance:

$$\int_{a}^{b} \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx = \left[\frac{1}{3(b-a)}\left(x - \frac{a+b}{2}\right)^3\right]_{a}^{b} = \frac{(b-a)^2}{12}$$

2. Let X be a continous random variable with a pdf

$$f(x) = ae^{-2|x|} = \begin{cases} ae^{-2x} & x \geq 0 \\ ae^{2x} & x < 0 \end{cases}$$

for some constant a.

   (a) Find a.
   (b) Find cdf of X.
   (c) Find mean of X. (You can guess without doing any computation.)
   (d) Find variance of X.

(e) What are the median and mode of X?

This is called Laplace random variable.

(a) To be a pdf, the integral ovr all real line should be 1. Hence

$$1 = \int_{-\infty}^{\infty} ae^{-2|x|}dx = \int_{-\infty}^{0} ae^{2x}dx + \int_{0}^{\infty} ae^{-2x}dx = a\left[\frac{1}{2}e^{2x}\right]_{-\infty}^{0} + a\left[-\frac{1}{2}e^{-2x}\right]_{0}^{\infty} = a$$

and $a = 1$.

(b) Compute by divide into two cases: $x \leq 0$ and $x > 0$. For the latter case, you can split the integral as $\int_{-\infty}^{x} = \int_{-\infty}^{0} + \int_{0}^{x}$. This gives

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} \frac{1}{2}e^{2x} & x \leq 0 \\ 1 - \frac{1}{2}e^{-2x} & x > 0 \end{cases}$$

(c) 0. You can do the integral $\int_{-\infty}^{\infty} xe^{-2|x|}dx = \int_{-\infty}^{0} xe^{2x}dx + \int_{0}^{\infty} xe^{-2x}dx$ directly (using integration by parts), but you can simply observe that the function $xf(x)$ is an odd function and the integral should be 0.

(d)

$$\int_{-\infty}^{\infty} x^2e^{-2|x|}dx = 2\int_{0}^{\infty} x^2e^{-2x}dx = \frac{1}{2}.$$

(e) It is 0, since $f(x)$ takes its maximum value at $x = 0$.

3. A type of lightbulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$.
   (a) What is the probability that a bulb fails within the first 200 hours?
   (b) What is the probability that a bulb burns for more than 800 hours?

The pdf is $f(x) = \frac{1}{\mu}e^{-x/\mu} = \frac{1}{1000}e^{-x/1000}$ for $x \geq 0$.

(a) $P(X \leq 200) = \int_0^{200} f(x)dx = [e^{-x/1000}]_0^{200} = 1 - e^{-1/5}$.

(b) $P(X \geq 800) = \int_{800}^{\infty} f(x)dx = [e^{-x/1000}]_{800}^{\infty} = e^{-4/5}$.

4. (a) Find mean and median of the pdf: $f(x) = 1/x^2$ for $1/2 \leq x \leq 1$ and $f(x) = 0$ otherwise.
   (b) Find mean and median of the *cdf*: $F(t) = t^2/4$ for $0 \leq t \leq 2$.

(a)
- mean: $\int_{1/2}^{1} x \cdot 1/x^2 dx = \int_{1/2}^{1} 1/x dx = [\ln x]_{1/2}^{1} = \ln 2$.
- median: it will be m if $\int_{1/2}^{m} x \cdot 1/x^2 dx = [\ln x]_{1/2}^{m} = \ln m + \ln 2 = 1/2$, so $\ln m = \frac{1}{2} - \ln 2 = \ln(e^{1/2}/2) \Rightarrow m = e^{1/2}/2$.

(b) First, the corresponding pdf is $F'(x) = x/2$ for $0 \leq x \leq 2$, and 0 otherwise. Then the mean is $\int_0^2 x \cdot x/2 dx = [x^3/6]_0^2 = 4/3$. For the median, it is the value m with $F(m) = m^2/4 = 1/2$, so $m = \sqrt{2}$.