1. Let

$$f(x) = \begin{cases} cx^2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

- For what value of c is f a valid PDF?
- For the value of c found above, find $P(X \ge 1/2)$.

(a) We need $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. For $0 \le x \le 1$, $x^2(1-x) \ge 0$, so $c \ge 0$. Now setting

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 cx^2(1-x)dx = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 = c\left[\frac{1}{12}\right]$$

we see $c = 12$.

(b)

$$\begin{aligned} \int_{1/2}^{\infty} f(x)dx &= \int_{1/2}^{1} 12x^2(1-x)dx \\ &= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_{1/2}^{1} \\ &= 12\left[\frac{1}{3} - \frac{1}{4} - \frac{1}{24} + \frac{1}{32}\right] \end{aligned}$$

2. Let $f(x) = \ln(x)$ for $1 \le x \le e$. Find $P(2 \le X \le e)$.

(a)

$$\begin{aligned} P(2 \le X \le e) &= \int_2^e f(x)dx \\ &= \int_2^e \ln(x) \\ &= x\ln(x) - x\Big|_2^e \\ &= e - e + 2\ln(2) - 2 \\ &= 2(\ln(2) - 1) \end{aligned}$$

3. Given the CDF $F(x) = \frac{1}{1+e^{-x}}$ for $x \in R$, find the PDF $f(x)$.

(a) A PDF is recovered from a CDF by taking the derivative.

$$f(x) = \frac{d}{dx}(1+e^{-x})^{-1} = -(1+e^{-x})^{-2}(-e^{-x}) = \frac{e^{-x}}{(1+e^{-x})^2}$$

4. Given the CDF $F(t) = 1 - (\beta/t)^\alpha$ for $t \geq \beta$ where α and β are positive constants, find the PDF $f(t)$.

(a) We again obtain the PDF by taking the derivative

$$\frac{d}{dt}1 - (\beta/t)^\alpha = -\beta^\alpha(-\alpha t^{-\alpha-1}) = \frac{\beta^\alpha \alpha}{t^{\alpha+1}}$$

Thus the pdf is $\beta^\alpha \alpha / t^{\alpha+1}$ for $t \geq \beta$.

5. Calculate the mean and standard deviation for a random variable with the following associated PDF.

$$f(t) = \begin{cases} 2e^{-2t} & \text{for } 0 \leq t \leq 2 \\ 0 & \text{else} \end{cases}$$

(a) Well

$$\mu = \int_{-\infty}^{\infty} tf(t)dt = \int_0^2 2te^{-2t}dt$$

We integrate by parts. Let $u = t$ and $dv = 2e^{-2t}dt$. Then $du = dt$ and $v = -e^{-2t}$. So

$$\mu = -te^{-2t}\Big|_0^2 - \int_0^2 -e^{-2t}dt = -te^{-2t}\Big|_0^2 - \frac{e^{-2t}}{2}\Big|_0^2 = -2e^{-4} - \frac{e^{-4}}{2} + \frac{1}{2}$$

(b)

$$\begin{aligned} Var(X) &= \int_{0^2}(t-\mu)^2 f(t)dt \\ &= \int_0^2 2\left(t - \frac{5e^{-4}+1}{2}\right)^2 e^{-2t}dt \\ &= \left[2\int_0^2 t^2e^{-2t}dt\right] - \left[2(5e^{-4}+1)\int_0^2 te^{-2t}dt\right] + \left[2\left(\frac{5e^{-4}+1}{2}\right)^2\int_0^2 e^{-2t}dt\right] \end{aligned}$$

Now

$$\begin{aligned} \int_0^2 t^2e^{-2t}dt &= \frac{-t^2e^{-2t}}{2}\Big|_0^2 - \int_0^2 \frac{-2te^{-2t}}{2}dt \\ &= \frac{-t^2e^{-2t}}{2}\Big|_0^2 - \frac{te^{-2t}}{2}\Big|_0^2 + \frac{-e^{-2t}}{4}\Big|_0^2 \\ &= \frac{-13e^{-4}+1}{4} \end{aligned}$$

and

$$\int_0^2 te^{-2t}dt = \frac{-te^{-2t}}{2}\Big|_0^2 + \int_0^2 \frac{e^{-2t}}{2}dt = \frac{-5e^{-4}+1}{4}$$

and

$$\int_0^2 e^{-2t}dt = \frac{1-e^{-4}}{2}.$$

Hence

$$Var(X) = 2\left(\frac{-13e^{-4}+1}{4}\right) - 2(5e^{-4}+1)\left(\frac{-5e^{-4}+1}{4}\right) + 2\left(\frac{5e^{-4}+1}{2}\right)^2\left(\frac{1-e^{-4}}{2}\right).$$

So the standard deviation is

$$\sigma = \sqrt{2\left(\frac{-13e^{-4}+1}{4}\right) - 2(5e^{-4}+1)\left(\frac{-5e^{-4}+1}{4}\right) + 2\left(\frac{5e^{-4}+1}{2}\right)^2\left(\frac{1-e^{-4}}{2}\right)}$$