1. It is well-known in Cookieland that cookie production is normally distributed. You randomly poll twenty-one citizens, and find that three of them make 5 cookies per year, four make 7 per year, seven make 8 per year, four make 9 per year, and three make 11 per year.

   (a) Approximate the mean and standard deviation of cookie production using the statistics you have learned.

   Solution: If you draw a histogram, you'll notice it is symmetric, with the center being 8; this is the mean. The sum-of-squares is $3(3^2)+4(1^2)+7(0^2)4(1^2)+3(3^2) = 62$. So the sampling variance is $\frac{62}{20} = 3.1$, and the sampling standard deviation is $\sqrt{3.1}$.

   (b) Then, give a 95% confidence interval for the actual mean, using $t_{95,20} \approx 2.09$.

   Solution: $(8 - 2.09\sqrt{3.1/21}, 8 + 2.09\sqrt{3.1/21})$.

2. Suppose that the maximum height in feet that a Lemon Demon plant reaches is a function of the daily water it receives (in gallons) and hours of music you play for it; we denote these as W and M respectively, with height $H = W + 2M$. Among a community of Lemon Demon plant growers, W and M are independent and approximately normally distributed, with $\mu_W = 2$, $\mu_M = 5$, $\sigma_W = 3$, and $\sigma_M = 2$. What is the pdf of H?

   Solution: H is normally distributed, with mean $\mu_H = 2 + 2 \cdot 5 = 12$. Its variance is $\text{Var}(H) = \text{Var}(W + 2M) = \text{Var}(W) + 4\text{Var}(M) = 3^2 + 4 \cdot (2^2) = 25$, so its standard deviation is 5.

3. We are going to prove that for a Poisson RV X with pdf $f(k) = \frac{e^{-\lambda}\lambda^k}{k!}$, it is indeed the case that $E[X] = \lambda$.

   (a) Write out the infinite series for $E[X] = \sum_{k=0}^{\infty} kf(k)$. Simplify it as much as you can.

Solution: $\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}$

(b) Since the zeroth term $0 \cdot f(0) = 0$, this sum basically starts from 1. We will perform an index change: letting $i = k - 1$, rewrite this sum in terms of i instead of k.

Solution: $\sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^{i+1}}{i!}$

(c) Factor out $\lambda e^{-\lambda}$, and pull it outside the sum.

Solution: $\lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$

(d) Using a Taylor series, it turns out that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use this to finally solve for the expected value.

Solution: $\lambda e^{-\lambda} e^{\lambda} = \lambda$.

4. Your company likes to occasionally offer free t-shirts to its customers. The number it gives out each day is approximately Poisson distributed, with $\lambda = 3$. You and your coworker like to bet on the number of free t-shirts offered each day. She thinks that at most 3 customers will win, and you say the opposite. Using $e^{-3} \approx 0.05$, what is the chance that you win?

   Solution: This is the complement of $\sum_{k=0}^{3} \frac{e^{-3}3^k}{k!}$. Factoring out the e^{-3} and replacing it with 0.05, we get $0.05(1 + 3 + \frac{9/2}{+}\frac{27/6}{)} = 0.05(13) = 0.65$. So your friend has a 65% chance to win, and you have a 35% chance.

5. (Optional) Recall that the pdf of a normal distribution with mean 0 and standard deviation σ is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

   (a) Find the second derivative of this function.

   Solution: Ignoring the coefficient for a second, the first derivative is $-\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$. The derivative of that is $-\frac{1}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} + \frac{x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}}$. Now reintroducing the constant coefficient, we get $\frac{1}{\sigma\sqrt{2\pi}}(-\frac{1}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} + \frac{x^2}{\sigma^4}e^{-\frac{x^2}{2\sigma^2}})$, which we can profitably rewrite as $(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2})f(x)$.

   (b) Set it equal to zero.

   Solution: Observe that f can never be 0, so this is really just setting $\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2} = 0$. Solving for x, then, gives $\pm\sigma$.

   (c) What does this tell you about the inflection points of a normal distribution?

Solution: As changing the mean just amounts to a horizontal translation of the graph, we see the inflection points of a normal distribution are always one standard deviation from the mean.