GSI : Seewoo Lee.

1. Consider the cyclic group $\mathbb{Z}_8$.
   (a) Find all generators of $\mathbb{Z}_8$.
   (b) For each generator, draw the corresponding Cayley graph with respect to the generating set consisting of that single generator.
2. Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(a, b) : a \in \mathbb{Z}_2, b \in \mathbb{Z}_4\}$.
   (a) What is the order of the group?
   (b) Is it cyclic? If so, find a generator. If not, explain why.
   (c) Find a generating set S of two elements and draw the corresponding Cayley graph.
3. Consider a cube with 8 vertices and 12 edges. Color each edge with three colors, where parallel edges are colored the same.
   (a) Show that this is a Cayley graph of a group G and a generating set $S \subseteq G$ (where all edges are considered bidirectional). What is the order of G and the size of S?
   (b) Is it abelian?
   (c) Show that every element of G has order at most 2.
4. Consider the dihedral group D_4 of a square.
   (a) What is the order of D_4? Describe all elements of D_4 in terms of symmetries of the square.
   (b) Let r be the counterclockwise rotation by $90°$. Let s be the reflection about the vertical axis. Write down all elements of D_4 in terms of r and s.
   (c) Let t be the reflection about the diagonal connecting the top left and bottom right vertices. Write down all elements of D_4 in terms of s and t.
   (d) Draw a Cayley graph of D_4 with respect to the generating set $\{r, s\}$.
5. Let $\mathrm{GL}_2(\mathbb{C})$ be the group of all invertible 2×2 matrices with complex entries. Let Q be the set of the following eight matrices:

$$\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \pm\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \quad \pm\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

   (a) *Be convinced* that Q is a group under matrix multiplication. (You do not need to check all the group axioms.)
   (b) Which of the matrices in Q have order 2? Which have order 4?
   (c) Let $a = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}0 & i\\ i & 0\end{smallmatrix}\right)$. Show that Q is generated by $\{a, b\}$ and draw the corresponding Cayley graph.
6. (*) Prove that the above groups are all possible groups of order 8, up to isomorphism. In other words, if G is a group of order 8, then it must be isomorphic to exactly one of the groups above.