GSI : Seewoo Lee.

1. (a) What are the possible group homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}$?
   (b) What are the possible group homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_6$?
   (c) Are $\mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{10}$ isomorphic?

2. Let $H$ and $K$ be groups, and let $G = H \times K$. Define the maps $f : H \to G$ and $g : G \to K$ by $f(h) = (h, e_K)$ and $g(h, k) = k$.
   (a) Show that $f$ is an injective homomorphism.
   (b) Show that $g$ is a surjective homomorphism.
   (c) Show that $f[H] = \ker(g)$.
   (d) Show that the converse does not hold in the following sense: Find groups $G, H, K$ and homomorphisms $f : H \to G$ and $g : G \to K$ such that $f$ is injective, $g$ is surjective, and $f[H] = \ker(g)$, but $G$ is *not* isomorphic to $H \times K$.

3. Let $G$ be a group generated by a subset $S \subseteq G$. Let $H$ be a subgroup of $G$. Prove that $H$ is a normal subgroup of $G$ if and only if for all $s \in S$, $sH = Hs$. In other words, it suffices to check the normality condition on the generators of $G$.

4. Consider the dihedral group $D_4$ of symmetries of a square. Let $r$ be a counterclockwise rotation by $90°$ and let $s$ be a reflection across a vertical axis of symmetry.
   (a) Find all elements of order 2 in $D_4$. There are 5 of them.
   (b) Consider a subgroup generated by $s$. What is the order of the subgroup? Is it normal?
   (c) Consider a subgroup generated by $r^2$. What is the order of the subgroup? Is it normal?
   (d) (Yes it is normal.) The factor group $D_4/\langle r^2 \rangle$ has order 4. You know there are two possible groups of order 4 up to isomorphism: $\mathbb{Z}_4$ and the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Which one is it?

5. Let $G = \mathrm{GL}_2(\mathbb{R})$. Consider the subset of invertible upper triangular matrices

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}$$

   (a) Show that $B$ is a subgroup of $G$ under matrix multiplication. Is it normal in $G$?
   (b) Show that the set of matrices

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \right\}$$

   is a normal subgroup of $B$.
   (c) Can you give a simple description of the factor group $B/N$?