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1. (a) Any homomorphism $\varphi : \mathbb{Z}_4 \to \mathbb{Z}$ satisfies $0 = \varphi(4 \cdot 1) = 4\varphi(1)$ in $\mathbb{Z}$, hence $\varphi(1) = 0$ and φ is trivial. Thus, only the zero homomorphism.

   (b) Homomorphisms $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_6$ are determined by $k = \varphi(1) \in \mathbb{Z}_6$ with $0 = \varphi(4 \cdot 1) = 4k$ in $\mathbb{Z}_6$. Since $4k \equiv 0 \pmod 6$ iff $k \in \{0, 3\}$, there are exactly two: the trivial map and the map sending $1 \mapsto 3$ (with kernel $\{0, 2\}$).

   (c) Yes. Using the structure theorem:

$$\mathbb{Z}_2 \times \mathbb{Z}_{20} \cong \mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_5) \cong \mathbb{Z}_4 \times (\mathbb{Z}_2 \times \mathbb{Z}_5) \cong \mathbb{Z}_4 \times \mathbb{Z}_{10}.$$

   One can write down an explicit isomorphism as well by tracking the elements through the isomorphisms above:

$$\varphi(a, b) = (b \bmod 4, (5a + 6b) \bmod 10)$$

   (Check that this is indeed a well-defined isomorphism.)

2. (a)

$$f(h_1h_2) = (h_1h_2, e_K) = (h_1, e_K)(h_2, e_K) = f(h_1)f(h_2),$$

   so f is a homomorphism. If $f(h) = (e_H, e_K)$, then $h = e_H$, so f is injective.

   (b)

$$g((h_1, k_1)(h_2, k_2)) = g((h_1h_2, k_1k_2)) = k_1k_2 = g((h_1, k_1))g((h_2, k_2)),$$

   so g is a homomorphism and for any $k \in K$, $g(e_H, k) = k$, so g is surjective.

   (c) $\ker(g) = \{(h, e_K) : h \in H\} = f[H]$.

   (d) Let $G = \mathbb{Z}_4$, $H = \mathbb{Z}_2$, $K = \mathbb{Z}_2$. Define $f : \mathbb{Z}_2 \to \mathbb{Z}_4$ by $f(1) = 2$ and $g : \mathbb{Z}_4 \to \mathbb{Z}_2$ by $g(x) = x \bmod 2$. Then f injective, g surjective, and $\operatorname{im}(f) = \{0, 2\} = \ker(g)$, but $G \not\cong H \times K$ since $\mathbb{Z}_4$ is cyclic whereas $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not.

3. Assume $sH = Hs$ for all $s \in S$. Since S generates G, any $g \in G$ can be written as a product of elements of S and their inverses. Note that if $sH = Hs$, then $s^{-1}H = Hs^{-1}$ as well. If $xH = Hx$ and $yH = Hy$, then

$$(xy)H = x(yH) = x(Hy) = (xH)y = (Hx)y = H(xy).$$

   By induction on length k of $g = s_1^{\pm} s_2^{\pm} \cdots s_k^{\pm}$, we conclude that $gH = Hg$ for any $g \in G$. Hence H is normal in G.

4. (a) Elements of order 2 in D_4: r^2 and the four reflections s, rs, r^2s, r^3s (total 5).

   (b) $\langle s \rangle = \{e, s\}$ has order 2. It is not normal: e.g., $rsr^{-1} \neq s$ (rsr^{-1} is a reflection across a horizontal axis), so $r\langle s \rangle r^{-1} \neq \langle s \rangle$.

   (c) $\langle r^2 \rangle = \{e, r^2\}$ has order 2. It is normal, and this can be checked by using the previous result: we have

$$rr^2r^{-1} = r^2, \quad sr^2s^{-1} = r^2,$$

   which are both in $\langle r^2 \rangle$. Since r and s generate D_4, we have $gr^2g^{-1} = r^2$ for any $g \in D_4$. Hence $\langle r^2 \rangle$ is normal in D_4.

   (d) In $D_4/\langle r^2 \rangle$, the images $\bar{r}, \bar{s}$ both have order 2, and the relation $srs = r^{-1}$ becomes $\bar{s}\,\bar{r}\,\bar{s} = \bar{r}$ (since $\bar{r}^{-1} = \bar{r}$), so $\bar{r}$ and $\bar{s}$ commute. Hence every element has order at most 2 and $D_4/\langle r^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

5. (a) Closure: for $g_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$ in B,

$$g_1g_2 = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ 0 & d_1d_2 \end{pmatrix} \in B$$

   since $aa' \neq 0$ and $dd' \neq 0$, so the product is invertible upper triangular. Identity is I. Inverse of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is $\begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix}$, which is upper triangular. Thus B is a group.

It is not normal in G; for example, take $g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G$ and $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in B$, then

$$ghg^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \notin B.$$

(b) For $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$ and $u = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in N$,

$$gug' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{d}n \\ 0 & 1 \end{pmatrix} \in N,$$

so N is normal in B.

(c) The map $\pi : B \to \mathbb{R}^\times \times \mathbb{R}^\times$, $\pi\big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\big) = (a, d)$, is a surjective homomorphism with kernel N. Hence $B/N \cong \mathbb{R}^\times \times \mathbb{R}^\times$.