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1. Let $g = (1, 0)$. The order of g in $G = \mathbb{Z}_4 \times \mathbb{Z}_6$ is 4. However, the order of $g + H$ in G/H is 2 since $2g = (2, 0) = (2, 6) \in H = \langle (1, 3) \rangle$. Thus the order of $g + H$ does not need to equal the order of g.

   In fact, the order of gH in G/H divides the order of g in G. Under the map $\phi : G \to G/H$ defined by $\phi(g) = gH$ (whose kernel is H), the image of the subgroup $\langle g \rangle$ is the subgroup $\langle gH \rangle$ in G/H. By the First Isomorphism Theorem, we have

   $$\langle g \rangle / (\langle g \rangle \cap H) \cong \langle gH \rangle .$$

   Thus, the order of gH, which is the order of the group on the right side, is given by

   $$|gH| = \frac{|g|}{|\langle g \rangle \cap H|},$$

2. (a) For $g = (2\,3)$, the fixed points are 1 and 4.

   (b) For $x = 1$, $G_x = \{\sigma \in S_4 : \sigma(1) = 1\} = \{e, (2\,3), (2\,4), (3\,4), (2\,3\,4), (2\,4\,3)\}$. This is isomorphic to S_3, as a permutation of $\{2, 3, 4\}$, and has order 6.

   (c) The action is transitive and $G \cdot 1 = \{1, 2, 3, 4\}$, so $|G \cdot 1| = 4 = \frac{24}{6} = \frac{|S_4|}{|S_3|}$

   (d) In general, S_n acts transitively on $\{1, \dots, n\}$; the stabilizer of 1 (or in fact, any i) is isomorphic to S_{n-1}. The orbit-stabilizer theorem gives $n = \frac{|S_n|}{|S_{n-1}|} = \frac{n!}{(n-1)!}$.

3. Note that the factor group is isomorphic to the image $\{\pm 1\}$, which is a group of order 2, so every element has order 1 or 2. The sign of

   $$\sigma = (1\,2\,3\,4\,5)(6\,7\,8)(9\,10)(11\,12\,13\,14)(15\,16\,17\,18\,19\,20)$$

   is $(+)(+)(-)(-)(-) = (-)$, so $\sigma \notin A_{100}$ and σA_{100} is the nontrivial element of the quotient. Hence the order is 2.

4. (a) With the left-action convention (apply s then r for rs):

   $$r = (1\,2\,3\,4\,5), \qquad s = (2\,5)(3\,4), \qquad rs = (1\,2)(3\,5).$$

   (b) $G_1 = \{e, s\}$, order 2.

   (c) $G \cdot 1 = \{1, 2, 3, 4, 5\}$. Basically, you can move 1 to any vertex by some rotation.

   (d) $|D_5| = 10 = |G \cdot 1| \cdot |G_1| = 5 \cdot 2$.

   (e) Since r and s generates D_5, you only need to check that $H = \langle r \rangle$ is fixed under the conjugation by r and s (Why?). Since $r \in H$, conjugation by r fixes H. Also, $srs^{-1} = r^{-1} \in H$, so conjugation by s also fixes H. Thus H is normal in D_5.

   Note that H has order 5 and index 2 in D_5. In general, any subgroup of index 2 is normal (Why?).

   (f) For general $n \geq 3$, stabilizer subgroup of any vertex has order 2, and the action is transitive on the n vertices, so the orbit has size n. We have $n = \frac{|D_n|}{|(D_n)_x|} = \frac{2n}{n}$. Also, the subgroup $\langle r \rangle \trianglelefteq D_n$ is normal (of order n and index 2).

5. To figure out the order of the group, let's label two adjacent vertices of the cube as 1 and 2. When we rotate the cube, 1 can be sent to any of the 8 vertices, and once we fix where 1 goes, 2 can be sent to any of the 3 adjacent vertices. Thus there are $8 \times 3 = 24$ possible rotations, so $|G| = 24$.

   In fact, the group is isomorphic to S_4. You can understand by regarding the rotations as permutations of the 4 long diagonals of the cube. This action is faithful and transitive, and the stabilizers are conjugate, yielding an isomorphism.