All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: multivariable function, domain, limit, continuity

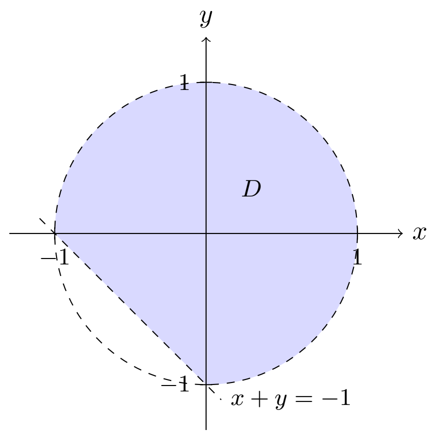
1. Find and sketch the domain of the following functions:
   (a) $f(x,y) = \frac{\ln(x+y+1)}{\sqrt{1-x^2-y^2}}$
   (b) $g(x,y,z) = \sqrt{1-x} + \sqrt{2-y} + \sqrt{3-z}$

2. Consider the function
$$f(x,y) = \begin{cases} \frac{x^3y}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
   (a) Show that the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = kx$ is 0 for any constant k.
   (b) Show that the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the parabola $y = kx^2$ is 0 for any constant k.
   (c) Show that the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the parabola $y = x^3$ is $\frac{1}{2}$.
   (d) Does the limit of $f(x,y)$ as $(x,y) \to (0,0)$ exist? Why or why not?

3. (a) Using the definition of limit, show that the function $f(x,y) = x + 2y$ is continuous at the point $(0,0)$.
   (b) (*) Using the definition of limit, for any a, b and c, show that the function $f(x,y) = ax^2+bxy+cy^2$ is continuous at the point $(0,0)$.

1. (a) We need $x+y+1>0$ (for ln) and $1-x^2-y^2>0$ (for $\sqrt{\ }$ in the denominator). So the domain is

$$D = \{(x,y) : x^2+y^2<1 \text{ and } x+y>-1\}.$$



(b) We need $1-x \geq 0$, $2-y \geq 0$, and $3-z \geq 0$. So the domain is

$$D = \{(x,y,z) \in \mathbb{R}^3 : x \leq 1,\ y \leq 2,\ z \leq 3\}.$$

2. (a) If $k \neq 0$: $f(x,kx) = \dfrac{x^3 \cdot kx}{x^6+k^2x^2} = \dfrac{kx^2}{x^4+k^2} \to \dfrac{0}{k^2} = 0$ as $x \to 0$. If $k=0$: $f(x,0)=0$. In both cases, the limit is 0.

(b) If $k \neq 0$: $f(x,kx^2) = \dfrac{x^3 \cdot kx^2}{x^6+k^2x^4} = \dfrac{kx}{x^2+k^2} \to \dfrac{0}{k^2} = 0$ as $x \to 0$. If $k=0$: $f(x,0)=0$. In both cases, the limit is 0.

(c) $f(x,x^3) = \dfrac{x^3 \cdot x^3}{x^6+(x^3)^2} = \dfrac{x^6}{2x^6} = \dfrac{1}{2}$ for all $x \neq 0$, so the limit is $\dfrac{1}{2}$.

(d) No. Along $y=0$ the limit is 0, but along $y=x^3$ the limit is $\frac{1}{2}$. Since different paths give different limits, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

3. (a) Let $\varepsilon>0$ and choose $\delta = \varepsilon/3$. If $\sqrt{x^2+y^2}<\delta$, then using $|x|,|y| \leq \sqrt{x^2+y^2}$,

$$|f(x,y)-f(0,0)| = |x+2y| \leq |x|+2|y| \leq 3\sqrt{x^2+y^2} < 3\delta = \varepsilon.$$

(b) (*) If $a=b=c=0$, then f is identically zero and continuity is trivial. Otherwise, let $M = |a|+|b|+|c|>0$. Since $|x|^2,|y|^2 \leq x^2+y^2$ and $|xy| \leq \sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2} = x^2+y^2$, we have

$$|f(x,y)-f(0,0)| = |ax^2+bxy+cy^2| \leq |a|x^2+|b||xy|+|c|y^2 \leq M(x^2+y^2).$$

Let $\varepsilon>0$ and choose $\delta=\sqrt{\varepsilon/M}$. If $\sqrt{x^2+y^2}<\delta$, then $x^2+y^2<\varepsilon/M$, so

$$|f(x,y)-f(0,0)| \leq M(x^2+y^2) < M\cdot\frac{\varepsilon}{M} = \varepsilon.$$