All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Partial derivatives, Tangent plane and approximations

1. Check that the following functions satisfy Clairaut's theorem $f_{xy} = f_{yx}$.
   (a) $f(x, y) = xe^{\sqrt{y}}$
   (b) $f(x, y) = \ln(\sin x + \cos y)$
2. (a) Is there a function $f(x, y)$ with
   $$f_x(x, y) = x + y, \quad f_y(x, y) = 2x + y$$
   (b) How about
   $$f_x(x, y) = x + 2y, \quad f_y(x, y) = 2x + y$$
3. Find an equation of the tangent plane to the given surface at the specified point.
   (a) $z = e^{2x-y}$, $(1, 2, 1)$
   (b) $x^2 + y^2 + z^2 = 9$, $(1, -2, 2)$

1. For each function, compute the mixed partial derivatives.

   (a) $f(x,y) = xe^{\sqrt{y}}$.

$$f_x = e^{\sqrt{y}}, \qquad f_y = \frac{x}{2\sqrt{y}}e^{\sqrt{y}}.$$

$$f_{xy} = \frac{\partial}{\partial y}\left(e^{\sqrt{y}}\right) = \frac{1}{2\sqrt{y}}e^{\sqrt{y}}, \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{y}}e^{\sqrt{y}}\right) = \frac{1}{2\sqrt{y}}e^{\sqrt{y}}.$$

   Hence $f_{xy} = f_{yx}$ (where these derivatives are defined, i.e. $y > 0$).

   (b) $f(x,y) = \ln(\sin x + \cos y)$.

$$f_x = \frac{\cos x}{\sin x + \cos y}, \qquad f_y = -\frac{\sin y}{\sin x + \cos y}.$$

$$f_{xy} = \cos x \cdot \frac{\sin y}{(\sin x + \cos y)^2} = \frac{\sin y \cos x}{(\sin x + \cos y)^2},$$

$$f_{yx} = -\sin y \cdot \left(-\frac{\cos x}{(\sin x + \cos y)^2}\right) = \frac{\sin y \cos x}{(\sin x + \cos y)^2}.$$

   Therefore $f_{xy} = f_{yx}$ (on $\sin x + \cos y > 0$).

2. (a) Suppose $f_x = x + y$ and $f_y = 2x + y$. Then

$$f_{xy} = \frac{\partial}{\partial y}(x+y) = 1, \qquad f_{yx} = \frac{\partial}{\partial x}(2x+y) = 2.$$

   Since $f_{xy} \neq f_{yx}$, such an f cannot exist.

   (b) If $f_x = x + 2y$ and $f_y = 2x + y$, then

$$f_{xy} = 2, \qquad f_{yx} = 2,$$

   so existence is possible. Integrate f_x with respect to x:

$$f(x,y) = \frac{x^2}{2} + 2xy + g(y).$$

   Differentiate with respect to y:

$$f_y = 2x + g'(y) = 2x + y \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C.$$

   One such function is

$$f(x,y) = \frac{x^2}{2} + 2xy + \frac{y^2}{2} + C.$$

3. (a) $z = e^{2x-y}$ at $(1,2,1)$.

$$f_x = 2e^{2x-y}, \qquad f_y = -e^{2x-y},$$

   so

$$f_x(1,2) = 2, \qquad f_y(1,2) = -1.$$

   Tangent plane:

$$z - 1 = f_x(1,2)(x-1) + f_y(1,2)(y-2) = 2(x-1) - (y-2),$$

   i.e.

$$z = 2x - y + 1.$$

(b) $x^2 + y^2 + z^2 = 9$ at $(1, -2, 2)$. View z as a function $z(x, y)$ and differentiate implicitly:

$$2x + 2z\,\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}, \qquad 2y + 2z\,\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

At $(x, y, z) = (1, -2, 2)$:

$$\frac{\partial z}{\partial x}(1, -2) = -\frac{1}{2}, \qquad \frac{\partial z}{\partial y}(1, -2) = 1.$$

So the tangent plane to $z = f(x, y)$ is

$$z - 2 = \frac{\partial z}{\partial x}(1, -2)(x - 1) + \frac{\partial z}{\partial y}(1, -2)(y + 2) = -\frac{1}{2}(x - 1) + (y + 2).$$

Equivalently,

$$x - 2y + 2z = 9.$$