All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Tangent planes and approximations, chain rule for partial derivatives, directional derivatives

1. Let
$$f(x, y) = xe^y \sin(xy) + x.$$
   (a) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 1)$.
   (b) Use (a) to approximate $f(1.03, -0.04)$.

2. Let
$$z = x^2y + \sin(xy), \qquad x = u + v^2, \qquad y = u^2 - v.$$
   Use the multivariable chain rule to compute $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ at $(u, v) = (1, 1)$.

3. Let
$$T(x, y, z) = xe^{yz}, \qquad x = t^2 + 1, \quad y = \sin t, \quad z = \cos t.$$
   Compute $\dfrac{dT}{dt}$ at $t = 0$.

4. Let
$$x^2y + yz + z^3 = 5.$$
   Find the equation of the tangent plane to this surface at the point $(1, 2, 1)$.

1. (a)
$$f_x = e^y \sin(xy) + xye^y \cos(xy) + 1, \qquad f_y = xe^y \sin(xy) + x^2e^y \cos(xy).$$
At $(1,0)$: $f(1,0) = 1,\ f_x(1,0) = 1,\ f_y(1,0) = 1$. Tangent plane: $z - 1 = 1(x-1) + 1(y-0)$, i.e. $z = x + y$.

   (b) By (a),
$$f(1.03, -0.04) \approx 1.03 + (-0.04) = 0.99.$$

2.
$$\frac{\partial z}{\partial x} = 2xy + y\cos(xy), \qquad \frac{\partial z}{\partial y} = x^2 + x\cos(xy).$$
At $(u,v) = (1,1)$, $(x,y) = (2,0)$, so
$$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = 6.$$
Also
$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 2, \quad \frac{\partial y}{\partial u} = 2, \quad \frac{\partial y}{\partial v} = -1.$$
Therefore
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 12, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -6.$$

3. $T_x = e^{yz},\ T_y = xze^{yz},\ T_z = xye^{yz}$. At $t = 0$: $(x,y,z) = (1,0,1)$ and $(x', y', z') = (0,1,0)$. So
$$\frac{dT}{dt} = T_x x' + T_y y' + T_z z' = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 = 1.$$

4. Treat z as a function of x, y in
$$x^2y + yz + z^3 = 5.$$
Differentiate implicitly with respect to x:
$$2xy + y\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$$
so
$$\frac{\partial z}{\partial x} = -\frac{2xy}{y + 3z^2}.$$
At $(1,2,1)$,
$$\frac{\partial z}{\partial x}(1,2) = -\frac{4}{5}.$$
Differentiate with respect to y:
$$x^2 + z + y\frac{\partial z}{\partial y} + 3z^2\frac{\partial z}{\partial y} = 0$$
so
$$\frac{\partial z}{\partial y} = -\frac{x^2 + z}{y + 3z^2}, \qquad \frac{\partial z}{\partial y}(1,2) = -\frac{2}{5}.$$
Therefore the tangent plane to $z = f(x,y)$ at $(1,2,1)$ is
$$z - 1 = \frac{\partial z}{\partial x}(1,2)(x-1) + \frac{\partial z}{\partial y}(1,2)(y-2)$$
i.e.
$$z - 1 = -\frac{4}{5}(x-1) - \frac{2}{5}(y-2).$$