All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Gradient and directional derivative

1. Let $f(x, y) = xe^{xy}$
   (a) Compute gradient of $f$.
   (b) Compute the directional derivative $D_{\mathbf{u}}f$ at $P = (1, 2)$ when $\mathbf{u} = \langle 1, -1\rangle$.
   (c) Find a unit vector $\mathbf{u}$ where $D_{\mathbf{u}}f$ at $P = (1, 2)$ is maximized.
   (d) Find unit vectors $\mathbf{u}$ where $D_{\mathbf{u}}f = 0$ at $P = (1, 2)$.
2. Let $T(x, y, z) = x^2y + yz$ and $P = (1, 2, -1)$:
   (a) Compute $\nabla T(P)$.
   (b) Compute $D_{\mathbf{u}}T(P)$ where $\mathbf{u}$ points from $P$ to $Q = (3, 1, 2)$.
   (c) Find a unit vector in the direction of steepest decrease at $P$.
3. At what point on the ellipsoid $x^2+y^2+2z^2 = 1$ is the tangent plane parallel to the plane $x+2y+2z = 1$?

1. (a) Partial derivatives of $f$ are

$$f_x = e^{xy} + xye^{xy}, \qquad f_y = x^2 e^{xy}.$$

So

$$\nabla f(x, y) = \left\langle e^{xy}(1 + xy),\ x^2 e^{xy} \right\rangle.$$

In particular, at $P = (1, 2)$,

$$\nabla f(1, 2) = e^2 \langle 3, 1 \rangle.$$

(b) Normalize $\langle 1, -1 \rangle$:

$$\mathbf{u} = \frac{\langle 1, -1 \rangle}{\sqrt{1^2 + (-1)^2}} = \frac{\langle 1, -1 \rangle}{\sqrt{2}}.$$

Then

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = e^2 \langle 3, 1 \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \frac{2e^2}{\sqrt{2}} = \sqrt{2}\, e^2.$$

(c) The maximum occurs in the gradient direction, so normalizing $\nabla f(1, 2)$ in (a) gives

$$\mathbf{u}_{\max} = \frac{\nabla f(1, 2)}{|\nabla f(1, 2)|} = \frac{\langle 3, 1 \rangle}{\sqrt{10}}.$$

(d) We need $\mathbf{u}$ that is orthogonal to $\nabla f(1, 2)$, so to $\langle 3, 1 \rangle$. The unit vectors are

$$\mathbf{u} = \pm \frac{\langle 1, -3 \rangle}{\sqrt{10}}.$$

2. (a) Partial derivatives of $T$ are

$$T_x = 2xy, \qquad T_y = x^2 + z, \qquad T_z = y.$$

So

$$\nabla T(1, 2, -1) = \langle 4, 0, 2 \rangle.$$

(b) The direction from $P$ to $Q$ is

$$\overrightarrow{PQ} = \langle 2, -1, 3 \rangle, \qquad \mathbf{u} = \frac{\langle 2, -1, 3 \rangle}{|\langle 2, -1, 3 \rangle|} = \frac{\langle 2, -1, 3 \rangle}{\sqrt{14}}.$$

Then

$$D_{\mathbf{u}} T(P) = \nabla T(P) \cdot \mathbf{u} = \langle 4, 0, 2 \rangle \cdot \frac{\langle 2, -1, 3 \rangle}{\sqrt{14}} = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

(c) Steepest decrease is in direction $-\nabla T(P)$:

$$\mathbf{u}_{\min} = -\frac{\nabla T(P)}{|\nabla T(P)|} = -\frac{\langle 4, 0, 2 \rangle}{\sqrt{20}} = \frac{\langle -2, 0, -1 \rangle}{\sqrt{5}}.$$

3. Let

$$F(x, y, z) = x^2 + y^2 + 2z^2 - 1.$$

The normal vector for the tangent plane at $(x, y, z)$ is

$$\nabla F = \langle 2x, 2y, 4z \rangle.$$

For parallel planes, this must be parallel to $\langle 1, 2, 2 \rangle$, so

$$\langle 2x, 2y, 4z \rangle = \lambda \langle 1, 2, 2 \rangle.$$

Hence

$$x = \frac{\lambda}{2}, \qquad y = \lambda, \qquad z = \frac{\lambda}{2}.$$

Substitute into $x^2 + y^2 + 2z^2 = 1$:

$$\frac{\lambda^2}{4} + \lambda^2 + 2 \cdot \frac{\lambda^2}{4} = 1 \Rightarrow \frac{7}{4}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{2}{\sqrt{7}}.$$

Therefore the points are

$$\left(\frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), \qquad \left(-\frac{1}{\sqrt{7}}, -\frac{2}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right).$$