All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Maximum and minimum of multivariable functions, Hessian matrix

1. Find all critical points of $f(x, y) = x^3 + y^3 - 3xy$ and classify each as a local maximum, local minimum, or saddle point.
2. Find the absolute maximum and minimum values of $f(x, y) = x^2 - 2xy + 2y$ on the closed triangular region $D$ with vertices $(0, 0)$, $(4, 0)$, and $(0, 2)$.
3. Find the point on the plane $2x - y + z = 1$ that is closest to the origin.
4. Find and classify all critical points of $f(x, y) = e^x(x + y^2 + 2y)$.

1. Setting $f_x = 3x^2 - 3y = 0$ and $f_y = 3y^2 - 3x = 0$ gives $y = x^2$ and $x = y^2$. Substituting, $x = x^4$, so $x(x^3 - 1) = 0$, giving critical points $(0, 0)$ and $(1, 1)$. Second partials: $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$.
   - At $(0, 0)$: $D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 9 = -9 < 0 \Rightarrow$ saddle point.
   - At $(1, 1)$: $D = (6)(6) - 9 = 27 > 0$ and $f_{xx} = 6 > 0 \Rightarrow$ local minimum.

2. Interior: $f_x = 2x - 2y = 0$ and $f_y = -2x + 2 = 0$ give $(x, y) = (1, 1)$, which lies inside $D$. $f(1, 1) = 1$.

   Boundary:
   - $y = 0$, $0 \le x \le 4$: $f = x^2$. min $f = 0$ at $(0, 0)$, max $f = 16$ at $(4, 0)$.
   - $x = 0$, $0 \le y \le 2$: $f = 2y$. min $f = 0$ at $(0, 0)$, max $f = 4$ at $(0, 2)$.
   - $x + 2y = 4$ (i.e. $x = 4 - 2y$), $0 \le y \le 2$: $f = 8y^2 - 22y + 16$. Setting $f'(y) = 16y - 22 = 0$ gives $y = 11/8$, $x = 5/4$, and $f = 7/8$ (minimum along the line segment).

   Comparing all values: absolute min $= 0$ at $(0, 0)$, absolute max $= 16$ at $(4, 0)$.

3. Substitute $z = 1 - 2x + y$ into the squared distance $f(x, y) = x^2 + y^2 + z^2$:

   $$f(x, y) = x^2 + y^2 + (1 - 2x + y)^2 = 5x^2 + 2y^2 - 4xy - 4x + 2y + 1.$$

   Setting $f_x = 10x - 4y - 4 = 0$ and $f_y = 4y - 4x + 2 = 0$: from the second equation, $y = x - \frac{1}{2}$. Substituting, $10x - 4(x - \frac{1}{2}) - 4 = 0 \Rightarrow 6x = 2 \Rightarrow x = \frac{1}{3}$, $y = -\frac{1}{6}$, $z = \frac{1}{6}$.

   Second partials: $f_{xx} = 10$, $f_{yy} = 4$, $f_{xy} = -4$. $D = (10)(4) - (-4)^2 = 24 > 0$ and $f_{xx} = 10 > 0$, so this is a local (and absolute) minimum. The closest point is $\left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$ with distance $\frac{1}{\sqrt{6}}$.

4. $f_x = e^x(x + 1 + y^2 + 2y)$ and $f_y = e^x(2y + 2)$. Since $e^x \ne 0$, setting $f_y = 0$ gives $y = -1$. Then $f_x = 0$ gives $x + 1 + 1 - 2 = 0$, so $x = 0$. The only critical point is $(0, -1)$.

   Second partials: $f_{xx} = e^x(x + 2 + y^2 + 2y)$, $f_{yy} = 2e^x$, $f_{xy} = e^x(2y + 2)$. At $(0, -1)$: $f_{xx} = 1$, $f_{yy} = 2$, $f_{xy} = 0$. $D = (1)(2) - 0 = 2 > 0$ and $f_{xx} = 1 > 0 \Rightarrow$ local minimum, with $f(0, -1) = -1$.