All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Lagrange multipliers

1. Find the maximum and minimum values of
$$f(x,y) = e^{xy}$$
subject to
$$x^2 + y^2 = 10.$$

2. Find the maximum and minimum values of
$$f(x,y) = x^2 + 2y^2$$
on the region
$$x^2 + 4y^2 \le 4.$$

3. A rectangular box with positive side lengths x, y, z has fixed volume 64. Find the dimensions that minimize its total surface area.

4. (*) With two constraints, find the maximum and minimum values of
$$f(x,y,z) = xyz$$
subject to
$$x^2 + y^2 + z^2 = 12, \qquad x + y + z = 0.$$

1. Since e^t is strictly increasing, this is equivalent to optimizing xy on the same constraint. Let $g(x,y) = x^2+y^2-10$. Lagrange equations:

$$(ye^{xy}, xe^{xy}) = \lambda(2x, 2y) \Rightarrow ye^{xy} = 2\lambda x,\ xe^{xy} = 2\lambda y.$$

Divide by $e^{xy} > 0$ to get $y = 2\mu x,\ x = 2\mu y$ (where $\mu = \lambda e^{-xy}$). So $1 = 4\mu^2$, hence $\mu = \pm\frac{1}{2}$. If $\mu = \frac{1}{2}$, then $y = x$ and $2x^2 = 10$, so $(x,y) = (\pm\sqrt{5}, \pm\sqrt{5})$ and $xy = 5$. If $\mu = -\frac{1}{2}$, then $y = -x$, so $(x,y) = (\pm\sqrt{5}, \mp\sqrt{5})$ and $xy = -5$. Therefore

$$f_{\max} = e^5, \qquad f_{\min} = e^{-5}.$$

2. Interior: $\nabla f = (2x, 4y) = (0,0)$ gives $(x,y) = (0,0)$, which is inside the region. Its value is $f(0,0) = 0$. On the boundary $x^2+4y^2 = 4$, let $g(x,y) = x^2+4y^2$. Lagrange equations:

$$(2x, 4y) = \lambda(2x, 8y).$$

So

$$x(1-\lambda) = 0, \qquad y(1-2\lambda) = 0.$$

Hence $x = 0$ or $y = 0$ (the case $x, y \neq 0$ would force $\lambda = 1$ and $\lambda = \frac{1}{2}$, impossible). On the boundary:

$$x = 0 \Rightarrow y = \pm 1, \qquad y = 0 \Rightarrow x = \pm 2.$$

Values:

$$f(0, \pm 1) = 2, \qquad f(\pm 2, 0) = 4.$$

Compare with interior value $f(0,0) = 0$:

$$f_{\min} = 0 \text{ at } (0,0), \qquad f_{\max} = 4 \text{ at } (\pm 2, 0).$$

3. Let $S = 2xy + 2xz + 2yz$ and $g(x,y,z) = xyz$. Then

$$(2y+2z, 2x+2z, 2x+2y) = \lambda(yz, xz, xy).$$

So

$$2(y+z) = \lambda yz, \quad 2(x+z) = \lambda xz, \quad 2(x+y) = \lambda xy.$$

Divide first two equations:

$$\frac{y+z}{x+z} = \frac{y}{x} \Rightarrow x(y+z) = y(x+z) \Rightarrow x = y.$$

Similarly, $x = z$. Thus $x = y = z$, and from $xyz = 64$:

$$x = y = z = 4.$$

Therefore the minimum surface area occurs for the cube $4\times 4\times 4$, with

$$S_{\min} = 2(16+16+16) = 96.$$

4. Let $g = x^2+y^2+z^2-12$ and $h = x+y+z$. Use

$$\nabla f = \lambda \nabla g + \mu \nabla h: \quad yz = 2\lambda x + \mu,\ xz = 2\lambda y + \mu,\ xy = 2\lambda z + \mu.$$

Subtract pairwise:

$$(y-x)(z+2\lambda) = 0,\ (z-y)(x+2\lambda) = 0,\ (x-z)(y+2\lambda) = 0.$$

If x, y, z were all distinct, then $x = y = z = -2\lambda$, impossible with $x+y+z = 0$ and $x^2+y^2+z^2 = 12$. So at least two variables are equal. Set $x = y = a$, then $z = -2a$ from $x+y+z = 0$. Using $x^2+y^2+z^2 = 12$:

$$6a^2 = 12 \Rightarrow a = \pm\sqrt{2}.$$

Candidate points are

$$(\sqrt{2}, \sqrt{2}, -2\sqrt{2}),\ (\sqrt{2}, -2\sqrt{2}, \sqrt{2}),\ (-2\sqrt{2}, \sqrt{2}, \sqrt{2}),$$
$$(-\sqrt{2}, -\sqrt{2}, 2\sqrt{2}),\ (-\sqrt{2}, 2\sqrt{2}, -\sqrt{2}),\ (2\sqrt{2}, -\sqrt{2}, -\sqrt{2}).$$

Their function values are

$$-4\sqrt{2} \quad \text{and} \quad 4\sqrt{2}.$$

Therefore:

$$f_{\max} = 4\sqrt{2} \text{ at } (2\sqrt{2}, -\sqrt{2}, -\sqrt{2}),\ (-\sqrt{2}, 2\sqrt{2}, -\sqrt{2}),\ (-\sqrt{2}, -\sqrt{2}, 2\sqrt{2}),$$
$$f_{\min} = -4\sqrt{2} \text{ at } (-2\sqrt{2}, \sqrt{2}, \sqrt{2}),\ (\sqrt{2}, -2\sqrt{2}, \sqrt{2}),\ (\sqrt{2}, \sqrt{2}, -2\sqrt{2}).$$