All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Multiple integral

1. Evaluate
$$\int_0^2 \int_{-1}^1 (2x - y + 3)\, dy\, dx.$$

2. Show that integrating in either order gives the same value for
$$\iint_R (e^x + \cos y)\, dA, \qquad R = [0,1] \times \left[0, \frac{\pi}{2}\right].$$
Compute it as $dx\, dy$ (integrate x first) and as $dy\, dx$ (integrate y first).

3. Find the volume under the surface
$$z = 8 - x^2 - y$$
and above the rectangular region
$$R = [0,2] \times [0,1].$$

4. Let
$$D = \{(x,y) : y \geq 0,\ x + y \leq 1,\ y \leq x + 1\}.$$
Evaluate
$$\iint_D (2x + 3y + 1)\, dA$$
in both descriptions of D: Type I and Type II.

5. Find the volume of the solid bounded by the surfaces
$$-1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad z \geq 0, \quad z \leq 3 - x - y.$$
Can you do it without doing any integration, but with geometric reasoning?

1.
$$\int_0^2\int_{-1}^1 (2x-y+3)\,dy\,dx = \int_0^2 \left[(2x+3)y-\frac{y^2}{2}\right]_{-1}^1 dx = \int_0^2 (4x+6)\,dx = \left[2x^2+6x\right]_0^2 = 20.$$

2. First as $dx\,dy$ (integrate x first):
$$\int_0^{\pi/2}\int_0^1 (e^x+\cos y)\,dx\,dy = \int_0^{\pi/2} [e^x + x\cos y]_0^1\,dy = \int_0^{\pi/2}(e-1+\cos y)\,dy = \frac{\pi}{2}(e-1)+1.$$
Now as $dy\,dx$ (integrate y first):
$$\int_0^1\int_0^{\pi/2}(e^x+\cos y)\,dy\,dx = \int_0^1 [e^x y + \sin y]_0^{\pi/2}\,dx = \int_0^1\left(\frac{\pi}{2}e^x+1\right)dx = \frac{\pi}{2}(e-1)+1.$$
Both orders agree, as expected.

3. The volume is
$$V = \iint_R (8-x^2-y)\,dA = \int_0^2\int_0^1 (8-x^2-y)\,dy\,dx.$$
Compute:
$$V = \int_0^2\left[(8-x^2)y-\frac{y^2}{2}\right]_0^1 dx = \int_0^2\left(\frac{15}{2}-x^2\right)dx = \left[\frac{15}{2}x-\frac{x^3}{3}\right]_0^2 = 15-\frac{8}{3} = \frac{37}{3}.$$

4. The region is bounded by $y=0$, $y=1-x$, and $y=x+1$ (triangle with vertices $(-1,0),(0,1),(1,0)$). For Type I, split at $x=0$:
$$\iint_D (2x+3y+1)\,dA = \int_{-1}^0\int_0^{x+1}(2x+3y+1)\,dy\,dx + \int_0^1\int_0^{1-x}(2x+3y+1)\,dy\,dx.$$
Compute:
$$= \int_{-1}^0\left(\frac{7}{2}x^2+6x+\frac{5}{2}\right)dx + \int_0^1\left(-\frac{1}{2}x^2-2x+\frac{5}{2}\right)dx = \frac{2}{3}+\frac{4}{3} = 2.$$
For Type II, $0\le y\le 1$ and $y-1\le x\le 1-y$, so
$$\int_0^1\int_{y-1}^{1-y}(2x+3y+1)\,dx\,dy = \int_0^1 (2+4y-6y^2)\,dy = 2.$$
They give the same result.

5. Let $R=[-1,1]\times[-1,1]$. The volume is
$$V = \iint_R (3-x-y)\,dA = 3\iint_R 1\,dA - \iint_R x\,dA - \iint_R y\,dA.$$
On the square $[-1,1]\times[-1,1]$, the x and y terms integrate to 0 by symmetry, so
$$V = 3\cdot \text{Area}([-1,1]\times[-1,1]) = 3\cdot 4 = 12.$$
Without integration: opposite points (x,y) and $(-x,-y)$ have heights
$$(3-x-y)+(3+x+y)=6,$$
so the average height over the square is 3. Hence the solid has the same volume as a rectangular rod (prism) with base area 4 and height 3, giving
$$V=12.$$