All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: multiple integral, normal line of a surface

1. Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $x + 2y + 3z = 6$.
2. Evaluate the integrals by reversing the order of integration:
   (a)
   $$\int_0^1 \int_{x^2}^1 \sqrt{y} \cos y dy dx$$
   (b)
   $$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} dy dx$$
3. Find the equation of the normal line to the surface $z = x^2 + y^2 - 3$ at the point $(1, 2, 2)$. Find the other point where the normal line intersects the surface.
4. Evaluate
   $$\iint_D e^{x+y}\, dA, \qquad D = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$
5. (BoC) Find the volume of the solid enclosed by three cylinders $x^2+y^2 = 1$, $y^2+z^2 = 1$, and $z^2+x^2 = 1$.

1. Write the plane as
$$z = \frac{6 - x - 2y}{3}.$$
The projection on the $xy$-plane is
$$R = \{(x, y) : x \geq 0,\ y \geq 0,\ x + 2y \leq 6\},$$
so
$$0 \leq x \leq 6, \qquad 0 \leq y \leq \frac{6 - x}{2}.$$
Hence
$$V = \iint_R z\, dA = \int_0^6 \int_0^{(6-x)/2} \frac{6 - x - 2y}{3}\, dy\, dx.$$
Compute:
$$V = \int_0^6 \frac{(6 - x)^2}{12}\, dx = \frac{1}{12}\int_0^6 (6 - x)^2\, dx = \frac{1}{12}\int_0^6 u^2\, du = 6.$$
Note that the same setup for $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant gives
$$V = \int_0^a \int_0^{b(1-x/a)} c\Big(1 - \frac{x}{a} - \frac{y}{b}\Big)\, dy\, dx = \frac{abc}{6},$$
which is the general tetrahedron formula.

2. (a) The region is
$$R = \{(x, y) : 0 \leq x \leq 1,\ x^2 \leq y \leq 1\}.$$
Reversing order:
$$0 \leq y \leq 1, \qquad 0 \leq x \leq \sqrt{y}.$$
Hence
$$\int_0^1 \int_{x^2}^1 \sqrt{y} \cos y\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y} \cos y\, dx\, dy = \int_0^1 y \cos y\, dy.$$
Integrate by parts:
$$\int_0^1 y \cos y\, dy = [y \sin y + \cos y]_0^1 = \sin 1 + \cos 1 - 1.$$
(b) The region is
$$R = \{(x, y) : 0 \leq x \leq 1,\ \sqrt{x} \leq y \leq 1\}.$$
Reversing order:
$$0 \leq y \leq 1, \qquad 0 \leq x \leq y^2.$$
So
$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx = \int_0^1 \int_0^{y^2} \sqrt{y^3 + 1}\, dx\, dy = \int_0^1 y^2 \sqrt{y^3 + 1}\, dy.$$
Let $u = y^3 + 1$, $du = 3y^2\, dy$:
$$\int_0^1 y^2 \sqrt{y^3 + 1}\, dy = \frac{1}{3}\int_1^2 u^{1/2}\, du = \frac{2}{9}\left[u^{3/2}\right]_1^2 = \frac{2}{9}(2\sqrt{2} - 1).$$

3. Write the surface as the level surface
$$F(x, y, z) = x^2 + y^2 - z - 3 = 0.$$
A normal vector at $(1, 2, 2)$ is
$$\nabla F(1, 2, 2) = (2x, 2y, -1)\Big|_{(1,2,2)} = (2, 4, -1).$$
Therefore the normal line is
$$(x, y, z) = (1, 2, 2) + t(2, 4, -1),$$
i.e.
$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 2 - t.$$

To find the other intersection with $z = x^2 + y^2 - 3$, substitute the line:

$$2 - t = (1+2t)^2 + (2+4t)^2 - 3 = 2 + 20t + 20t^2.$$

So

$$20t^2 + 21t = 0 \quad \Longrightarrow \quad t = 0 \text{ or } t = -\frac{21}{20}.$$

$t = 0$ is the given point, so the other point is at $t = -\frac{21}{20}$:

$$\left(1 + 2\left(-\frac{21}{20}\right),\ 2 + 4\left(-\frac{21}{20}\right),\ 2 - \left(-\frac{21}{20}\right)\right) = \left(-\frac{11}{10}, -\frac{11}{5}, \frac{61}{20}\right).$$

4. The region $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$ is a triangle, so

$$\iint_D e^{x+y}\, dA = \int_0^1 \int_0^{1-x} e^{x+y}\, dy\, dx.$$

Compute the inner integral:

$$\int_0^{1-x} e^{x+y}\, dy = e^x \int_0^{1-x} e^y\, dy = e^x \left(e^{1-x} - 1\right) = e - e^x.$$

Hence

$$\iint_D e^{x+y}\, dA = \int_0^1 (e - e^x)\, dx = [ex - e^x]_0^1 = 1.$$

5. The answer is $8(2 - \sqrt{2})$.