All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: double integral in polar coordinate

1. Compute the integral
$$\int_{\pi/4}^{\pi}\int_{2}^{3} r dr d\theta.$$
Can you compute it without doing integration?

2. Find the volume of the solid below the plane $x + y + 2z = 4$ and $x^2 + y^2 \leq 1$.

3. Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 4$.

4. Use polar coordinate to combine the sum
$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} x dy dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} x dy dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} x dy dx$$
into one double integral, then evaluate the double integral.

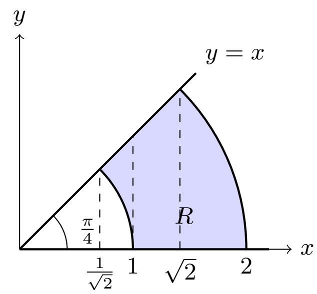
1. We have
$$\int_{\pi/4}^{\pi}\int_2^3 r\,dr\,d\theta = \int_{\pi/4}^{\pi}\left[\frac{1}{2}r^2\right]_2^3 d\theta = \left(\pi - \frac{\pi}{4}\right)\cdot\frac{5}{2} = \frac{15\pi}{8}.$$
Since $r\,dr\,d\theta = dA$, the integral is simply the area of an annular sector. So it can also be computed as
$$\frac{1}{2}\cdot 3^2\cdot\left(\pi-\frac{\pi}{4}\right) - \frac{1}{2}\cdot 2^2\cdot\left(\pi-\frac{\pi}{4}\right) = \frac{15\pi}{8}.$$

2. Let D be the unit disk, $x^2+y^2\le 1$. The volume is equal to
$$\iint_D \frac{4-x-y}{2}\,dA = \int_0^{2\pi}\int_0^1 \frac{4-r\cos\theta - r\sin\theta}{2}\,r\,dr\,d\theta.$$
Hence
$$\begin{aligned} V &= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(4r - r^2\cos\theta - r^2\sin\theta\right)\,dr\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left[2-\frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right]\,d\theta \\ &= \frac{1}{2}\left(4\pi - \frac{1}{3}\int_0^{2\pi}\cos\theta\,d\theta - \frac{1}{3}\int_0^{2\pi}\sin\theta\,d\theta\right) \\ &= 2\pi. \end{aligned}$$
By symmetry, $\iint_D x\,dA = \iint_D y\,dA = 0$, so one can also see immediately that
$$V = \frac{1}{2}\iint_D 4\,dA = 2\pi.$$

3. For the sphere, we have
$$x^2+y^2+z^2 \Leftrightarrow z = \pm\sqrt{9-r^2},$$
and outside the cylinder $x^2+y^2=4$ means $r=\sqrt{x^2+y^2}\ge 2$. Therefore
$$0\le\theta\le 2\pi,\qquad 2\le r\le 3,\qquad -\sqrt{9-r^2}\le z\le\sqrt{9-r^2}.$$
The volume is
$$\begin{aligned} V &= \int_0^{2\pi}\int_2^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_2^3 2r\sqrt{9-r^2}\,dr\,d\theta \\ &= 4\pi\int_2^3 r\sqrt{9-r^2}\,dr. \end{aligned}$$
Let $u = 9-r^2$, so $du = -2r\,dr$. Then
$$\begin{aligned} \int_2^3 r\sqrt{9-r^2}\,dr &= -\frac{1}{2}\int_5^0 u^{1/2}\,du = \frac{1}{2}\int_0^5 u^{1/2}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^5 = \frac{5\sqrt{5}}{3}. \end{aligned}$$
Therefore
$$V = 4\pi\cdot\frac{5\sqrt{5}}{3} = \frac{20\pi\sqrt{5}}{3}.$$

4. The three integrals describe the region in the first quadrant between the circles $r=1$ and $r=2$, and below the line $y=x$. So in polar coordinates the region is
$$R = \{(r,\theta) : 1\le r\le 2,\ 0\le\theta\le\tfrac{\pi}{4}\}.$$

A sketch of the region is shown below. The dashed vertical lines at $x = 1$ and $x = \sqrt{2}$ split R into the three Type I pieces appearing in the original sum.



Hence

$$1 \le r \le 2, \qquad 0 \le \theta \le \frac{\pi}{4}.$$

Since $x = r\cos\theta$ and $dA = r\,dr\,d\theta$, the sum becomes

$$\int_0^{\pi/4}\int_1^2 r\cos\theta \cdot r\,dr\,d\theta = \int_0^{\pi/4}\int_1^2 r^2\cos\theta\,dr\,d\theta.$$

Now evaluate:

$$\begin{aligned}\int_0^{\pi/4}\int_1^2 r^2\cos\theta\,dr\,d\theta &= \int_0^{\pi/4}\cos\theta\left[\frac{r^3}{3}\right]_1^2 d\theta\\ &= \frac{7}{3}\int_0^{\pi/4}\cos\theta\,d\theta\\ &= \frac{7}{3}\left[\sin\theta\right]_0^{\pi/4} = \frac{7}{3}\cdot\frac{\sqrt{2}}{2} = \frac{7\sqrt{2}}{6}.\end{aligned}$$